HYPERSURFACE INSERTION WINDOW FOR LONG TERM ORBITAL STABILITY OF ARTIFICIAL SATELLITES ABOUT THE PLANET VENUS

THESIS

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THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
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In Partial Fulfillment of the
Requirements for the Degree of
Master of Science in Astronautical Engineering

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Preface

The purpose of this study was to develop an analytical model for the six dimensional hypersurface above the planet Venus for a five year survivability for an artificial satellite.

Extensive modeling was made to try and map the contour of the hypersurface above Venus. This thesis gives just a breaking of the iceberg for survivability windows. There is a lot of further research to be performed in this field.

In evaluating the theoretical model and writing this thesis I have received a great deal of help from several people. I am greatly indebted to my faculty advisor, Capt. Rodney Bain, for his extensive help. Specifically, I owe him a large thanks for the mathematical knowledge he has tried to pass on to me. I would also like to thank my wife, [redacted] and my daughters; [redacted] They all have been an inspiration.

And, finally, I would like to thank the Most Awesome, Kind, Generous, Patient, Knowledgeable, Reigning Master of Sinanju, the Sun Source of all the Martial Arts, Master Chiun and his unworthy student, the Pale Piece of Pig's Ear, Remo [redacted] for the hours of escape they provided from the realities of AFIT.

Robert L. Dudley, Jr.

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Notation

Roman:

A'	satellite reference area
a	semi-major axis
a_e	semi-major axis of earth
a_p	semi-major axis of Venus
C_d	atmospheric drag coefficient
c_i	least squares constant
D	atmospheric drag
d	diameter of a satellite
e	eccentricity
F_{lmp}	inclination function
f	true anomaly
G_{lpq}	eccentricity function
g	metric tensor
H	density scale height
h	satellite altitude
h	equinoctial element
h_0	density reference altitude
h_p	periapsis altitude
i	inclination
j	imaginary number
Kn	Knudsen Number
k	equinoctial element
LPE	Lagrange's Planetary Equations
l	length of a satellite
M	mean anomaly
m	mass of the satellite
m	mass of a molecule
n	mean motion
n	molecular number
P	prime integer
Q	prime integer

R_i	least squares residual
R, R^*	perturbation
R_p	planet radius plus atmospheric blockage
r	planet radius
r	collision radius of a molecule
r	correlation coefficient
r_s	position of sun relative to Venus
r_{sun}	distance of sun at 1 A.U.
S	angle between the sun and the satellite
$S0$	ratio of P/Q
S_1, S_2, S_3	Venus centered coordinate system
T_i	original molecular temperature
T_{lmt}	coefficient of Associated Legendre Polynomial
T_r	re-emmited molecular temperature
T_s	satellite temperature
t	time
u	true anomaly plus argument of periapsis
V	gravitational potential energy
v	satellite velocity
$X_0^{n,m}$	Hansen's coefficient

<u>Greek:</u>

α	right ascension
α	reference density altitude
α	accommodation coefficient
β'	beta function of eccentricity
γ	black body radiation constant
δ	declination
θ	latitude
θ_g	prime meridian
λ	molecular mean free path
λ_N	stroboscopic mean node
π	3.141592654...

ρ	atmospheric density
ρ_0	atmospheric reference density
σ	molecular collision cross section
σ_{hp}	standard deviation of the periapsis altitude
ϕ	longitude
Ω	longitude of the ascending node
ω	argument of periapsis

Miscellaneous:

∇	gradient

Abstract

This study develops an analytic function for the six dimensional surface (or hypersurface) above the planet Venus for a five earth year survivability for an artificial satellite.

Current US policy concerning the exploration of other planets, via artificial satellites, requires the satellites be sterilized (5:61). This is a very time intensive and costly practice. Developing the ability to estimate the life time of an artificial satellite that can no longer perform its station keeping duties may allow the sterilization procedures before launch to be waived. The objective is to develop a five year survivability function (denoted by h_p) in the orbital parameter space above which a satellite has at least five years to survive before it impacts the planet's surface.

Perturbations effects which would cause the satellite's orbit to deteriorate are modeled and include: a) the geopotential of the planet, b) the effects of solar wind, and c) the drag on the satellite due to the atmosphere of the planet.

The model was then interpolated to provide an analytic function for five year survivability.

HYPERSURFACE INSERTION WINDOW FOR LONG TERM ORBITAL STABILITY OF ARTIFICIAL SATELLITES ABOUT THE PLANET VENUS

I. Introduction

Currently, the United States spends millions of dollars per satellite for sterilization prior to launch. There is a concern that, should an earth borne satellite impact another planet, the satellite would contaminate the indigenous environment. A likely scenario is a satellite in planetary orbit with its station keeping fuel depleted. Predicting the remaining life time of the satellite (i.e., the amount of time prior to the satellite entering the planet's atmosphere) could eliminate the need for sterilization and replace this practice with a rescue mission.

This thesis predicts the life time of a nonmaneuverable satellite by generating a five year hypersurface. This is a six dimensional surface, in the orbital parameters space, which marks the point of five (earth) years to impact. A satellite above this surface has five years to survive. Once a satellite contacts this surface there will be a specific amount of time to initiate a rescue (or destroy) mission.

Specifically, a perturbation model for a satellite orbiting Venus will be calculated using a 4x4 geopotential model for Venus, a solar wind

model, and a model for atmospheric drag. In order to decrease the size of the dimensional space the initial mean anomaly M and the initial argument of periapsis ω will be considered constant. This will reduce the space to four dimensions.

II. Model

In generating the model for this thesis, there were three perturbations considered: the geopotential of Venus, atmospheric drag, and solar wind. The perturbations are given in the following sections along with Lagrange's Planetary Equations (LPE) and a model of a typical satellite.

Lagrange's equations and the perturbations are given in terms of a hybrid set of parameters. For low eccentricities (on the order of 0.02 to 0.10) the eccentricity and the argument of perigee are replaced by two equinoctial elements, h and k (from a coordinate system with singularities at $i = \pi$ and for rectilinear orbits). Also, for long term effects of the orbit the short period mean anomaly M is replaced with the stroboscopic mean node (λ_N). This stroboscopic mean node acts as a discrete measurement of the variation of the mean anomaly at times Δt. The inclination, the longitude of the ascending node, and the semimajor axis are given typical values for an experimental satellite. This set of orbital parameters, along with their relationship to the classical set, will be explained in more detail in the next section.

Lagrange's Orbital Equations

Lagrange's Planetary Equations (LPE), written in terms of the orbital elements (a,e,i,Ω,ω,M), are (13:29):

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R^*}{\partial M} \tag{2.1.1a}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial R^*}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R^*}{\partial \omega} \tag{2.1.1b}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R^*}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R^*}{\partial e} \tag{2.1.1c}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R^*}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R^*}{\partial \Omega} \tag{2.1.1d}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R^*}{\partial i} \tag{2.1.1e}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial R^*}{\partial e} - \frac{2}{na}\frac{\partial R^*}{\partial a} \tag{2.1.1f}$$

Where R^* is the disturbing potential and n is the mean motion.

The LPE contain terms with the eccentricity and the inclination in the denominator. For the small values of these elements considered here ($0.02 \leq e \leq 0.10$ and $0.5 \leq i \leq 15.5$) a more well behaved set of variables will be used: two of the equinoctial elements, h and k. These elements eliminate the singularity due to zero eccentricity, and have their own singularity away from the range of interest at $i = \pi$ (22:23). h and k are given by

$$h = e\sin\omega \tag{2.1.2a}$$

$$k = e\cos\omega \qquad (2.1.2b)$$

Since only long term orbital behavior is of interest, the fast variable M (the Mean Anomaly) will be replaced with the stroboscopic mean node, λ_N(7:167-189). Resonance occurs when integer multiples of the mean secular rate of the various orbital parameters are matched with the coefficients of the geopotential. These resonance effects are due to longitude dependent tesserals in the central body gravity field. The stroboscopic mean node reveals the most pronounced effects of resonance. This allows us to retain any resonance effects due to tesseral harmonics (13,1966:49-56). The stroboscopic mean node is given by

$$\lambda_N = \frac{M+\omega}{S_0} + \left(\Omega - \theta_e\right) \qquad (2.1.3)$$

where S_0 is the ratio of two relatively prime integers, P/Q, approximating the number of nodal crossings per planet revolution. θ_e is the prime meridian angle. Gedeon (7:171) defines λ_N by considering a "mean satellite". Now, input $M + \Omega = 0$ at t=0. If Venus is illuminated with a strobe light the mean satellite will be seen above the equator at $\lambda_N = \Omega - \theta_e$ longitude. If Venus is flashed again after Q days, then (if $\lambda_N = 0$) the mean satellite will be at the same longitude. However, if $\lambda_N \neq 0$, the satellite will be a distance $\int \lambda_N dt$ away. This gives a measure of the change of the mean node from period to period.

The LPE must be written in terms of the new orbital parameter set (a,h,i,k,Ω,λ_N). First transform the disturbing potential R^*(a,e,i,Ω,ω,M) into a function of the new elements R(a,h,i,k,Ω,λ_N).

$$\frac{\partial R'}{\partial \omega} = k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial k} + \frac{1}{S_0}\frac{\partial R}{\partial \lambda_N} \tag{2.1.4a}$$

$$\frac{\partial R'}{\partial e} = \frac{h}{e}\frac{\partial R}{\partial h} + \frac{k}{e}\frac{\partial R}{\partial k} \tag{2.1.4b}$$

$$\frac{\partial R'}{\partial \Omega} = \frac{\partial R}{\partial \Omega} + \frac{\partial R}{\partial \lambda_N} \tag{2.1.4c}$$

$$\frac{\partial R'}{\partial M} = \frac{1}{S_0}\frac{\partial R}{\partial \lambda_N} \tag{2.1.4d}$$

The orbital equations for h, k, and λ_N are

$$\frac{dh}{dt} = \frac{h}{e}\frac{de}{dt} + k\frac{d\omega}{dt} \tag{2.1.5a}$$

$$\frac{dk}{dt} = \frac{k}{e}\frac{de}{dt} - h\frac{d\omega}{dt} \tag{2.1.5b}$$

$$\frac{d\lambda_N}{dt} = \frac{1}{S_0}\left(\frac{dM}{dt} + \frac{d\omega}{dt}\right) + \frac{d\Omega}{dt} - \frac{d\theta_g}{dt} \tag{2.1.5c}$$

Now rewrite the LPE into the new set of orbital parameters by way of the above equations(9:2) (15:1).

$$\frac{da}{dt} = \frac{2}{na}\frac{1}{S_0}\frac{\partial R}{\partial \lambda_N} \tag{2.1.6a}$$

$$\frac{dh}{dt} = \frac{\sqrt{1-e^2}}{na^2}\frac{\partial R}{\partial k} - \frac{k\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} - \frac{h\sqrt{1-e^2}}{na^2 S_0}\beta'\frac{\partial R}{\partial \lambda_N} \tag{2.1.6b}$$

$$\frac{di}{dt}=\frac{\cot i}{na^2\sqrt{1-e^2}\sin i}\left[k\frac{\partial R}{\partial h}-h\frac{\partial R}{\partial k}+\frac{1}{S_0}\frac{\partial R}{\partial \lambda_N}\right]$$

$$-\frac{1}{na^2\sqrt{1-e^2}\sin i}\left[\frac{\partial R}{\partial \Omega}+\frac{\partial R}{\partial \lambda_N}\right] \tag{2.1.6c}$$

$$\frac{dk}{dt}=\frac{\sqrt{1-e^2}}{na^2}\frac{\partial R}{\partial h}+\frac{h\cot i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}-\frac{k\sqrt{1-e^2}}{naS_0}\beta'\frac{\partial R}{\partial \lambda_N} \tag{2.1.6d}$$

$$\frac{d\Omega}{dt}=\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \tag{2.1.6e}$$

$$\frac{d\lambda_N}{dt}=\frac{n}{S_0}-\frac{d\theta_g}{dt}+\frac{1}{na^2S_0}\left(\sqrt{1-e^2}\beta'\left(h\frac{\partial R}{\partial h}+k\frac{\partial R}{\partial k}\right)\right.$$

$$\left.-2a\frac{\partial R}{\partial a}+\frac{S_0-\cos i}{\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}\right) \tag{2.1.6f}$$

where $e=\sqrt{h^2+k^2}$

$$\beta'=\frac{1}{1+\sqrt{1-e^2}}$$

Gravitational Potential Model

In order to get an expression for the gravitational potential (geopotential), first look at the Laplacian of the potential. Assume the geopotential to be conservative, then the Laplacian will be zero. In tensor notation

$$\nabla^2 V = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left[\sqrt{g} g^{kl} \frac{\partial V}{\partial x^l} \right] = 0 \tag{2.2.1}$$

where $g^{kl} =$ contravariant metric tensor

$$\sqrt{g} = [\det(g_{ij})]^{1/2}$$

Using spherical coordinates (radius(r), longitude(ϕ), and latitude(θ)), define the differential length as

$$(ds)^2 = (dr)^2 + (r\cos\theta d\phi)^2 + (rd\theta)^2$$

This will make the covariant metric tensor

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2\cos^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

and the contravariant metric tensor

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2\cos^2\theta} & 0 \\ 0 & 0 & \dfrac{1}{r^2} \end{pmatrix}$$

Now, to evaluate the Laplacian, note

$$\sqrt{g} = r^2\cos\theta$$

and

$$i \neq j \Rightarrow g^{ij} = 0$$

For $i=j=1$: $x^1 = r$ and

$$\frac{1}{r^2\cos\theta}\frac{\partial}{\partial r}\left[r^2\cos\theta\frac{\partial V}{\partial r}\right]=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) \tag{2.2.2a}$$

For $i=j=2$: $x^2 = \phi$ and

$$\frac{1}{r^2\cos\theta}\frac{\partial}{\partial\phi}\left[r^2\cos\theta\frac{1}{r^2\cos\theta}\frac{\partial V}{\partial\phi}\right]=\frac{1}{r^2\cos\theta}\frac{\partial^2 V}{\partial\phi^2} \tag{2.2.2b}$$

For $i=j=3$: $x^3 = \theta$ and

$$\frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left[r^2\cos\theta\frac{1}{r^2}\frac{\partial V}{\partial\theta}\right]=\frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial V}{\partial\theta}\right) \tag{2.2.2c}$$

Substituting eqn.s(2.2.2) into the Laplacian of the geopotential (eqn(2.2.1)) and assuming a conservative field yields

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)$$

$$+\frac{1}{r^2\cos\theta}\frac{\partial^2 V}{\partial\phi^2}+\frac{1}{r^2\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial V}{\partial\theta}\right)$$

$$\nabla^2 V=0 \tag{2.2.3}$$

Assuming the potential is linear and separable, then

$$V(r,\theta,\phi)=R(r)B(\theta)\Phi(\phi) \tag{2.2.4}$$

and substituting eqn(2.2.4) into eqn(2.2.3) yields

$$B\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+R\Phi\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)+RB\left(\frac{1}{\cos^2\theta}\frac{d^2\Phi}{d\phi^2}\right)=0 \quad (2.2.5)$$

Multiply the above equation by $\cos^2\theta/RB\Phi$ to get Laplace's equation in the form

$$\frac{\cos^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\cos\theta}{B}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0 \quad (2.2.6)$$

Since Φ is alone in Laplace's equation, it can be separated from the rest of the equation and both sides can be set equal to the separation constant $-m^2$, so

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0 \quad (2.2.7)$$

This is a Sturm-Liouville equation with the solution

$$\Phi(\phi)=C_m\cos m\phi+S_m\sin m\phi \text{ where } m=1,2,3,\ldots \quad (2.2.8)$$

Rewriting Laplace's equation

$$\frac{\cos^2\theta}{R}\left[\frac{d}{dr}\left(\frac{dR}{dr}\right)\right]+\frac{\cos\theta}{B}\left[\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)\right]=m^2$$

and dividing by $\cos^2\theta$ and separating, produces

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{m^2}{\cos^2\theta}-\frac{1}{B\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)=l \quad (2.2.9)$$

To evaluate $B(\theta)$, using eqn(2.2.9), first rewrite the right hand side as

$$\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)-\left(\frac{m^2}{\cos^2\theta}-l\right)B=0$$

Next transform the above equation to the form of a Legendre's Associated ordinary differential equation. Let $\sin\theta = x$ and $l = n(n+1)$ so

$$\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)+\left[n(n+1)\frac{m^2}{1-x^2}\right]B=0 \qquad (2.2.10)$$

where the identity $\cos^2\theta = 1 - \sin^2\theta = 1 - x^2$ was used.

Evaluating the derivatives yields

$$\frac{1}{\cos\theta}\frac{d}{d\theta}\left(\cos\theta\frac{dB}{d\theta}\right)=-2x\frac{dB}{dx}+(1-x)\frac{d^2B}{dx^2}$$

Substituting into the Associated Legendre's equation, eqn(2.2.10):

$$\left(1-x^2\right)\frac{d^2B}{dx^2}-2x\frac{dB}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]B=0 \qquad (2.2.11)$$

The solution to the Associated Legendre's equation is

$$B(\theta)=P_n^m(\sin\theta) \qquad (2.2.12a)$$

where $l=n(n+1)$

$$P_l^m(\sin\theta)=\cos^m\theta\sum_{t=0}^{|(l-m)/2|}T_{lmt}\sin^{l-m-2t}\theta \qquad (2.2.12b)$$

$$T_{lmt} = \frac{(-1)^t(2l-2t)!}{2^l t!(l-t)!(l-m-2t)!} \qquad (2.2.12c)$$

All that is left is to solve for $R(r)$. Looking at the potential expression where $B(\theta)$ was separated (eqn(2.2.9)) yields

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0 \qquad (2.2.13)$$

There are two possible solutions to the above equation. The first solution, $R(r) = r^n$, blows up as $r \to \infty$. Therefore, only keep the solution

$$R(r) = r^{-(n+1)} \qquad (2.2.14)$$

Combining the solutions for R, B, and Φ generates the following relationship for the geopotential (in terms of radius, longitude, and latitude). Note that the terms μ and R_e have been added. This is to make the C_{lm} and S_{lm} terms dimensionless.

$$V(r,\theta,\phi) = -\frac{\mu}{r}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{r}{R_e}\right)^{-l} P_l^m(\sin\theta)\left(C_{lm}\cos m\phi + S_{lm}\sin m\phi\right) \qquad (2.2.15)$$

Now translate the geopotential from its spherical harmonic representation (eqn(2.2.15)) to a Keplerian representation (see Appendix *A* for a full derivation). In order to do this, two new functions must be introduced: the inclination Function $F_{lmp}(i)$ and the Eccentricity Function $G_{lpq}(e)$.

The Inclination Function is defined as

$$F_{lmp}(i) = \sum_{t=0}^{(t)_{max}} \frac{(2l-2t)!}{t!(l-t)!(l-m-2t)!2^{2l-2t}} \sin^{(l-m-2t)}(i)$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^{s}(i) \sum_{g=g_L}^{g_U} \binom{l-m-2t+s}{g} \binom{m-s}{p-t-g} (-1)^{g-k} \qquad (2.2.16)$$

The Eccentricity Function may be defined as

$$G_{lp(2p-l)} = \frac{1}{a^{l+1}(1-e^2)^{l-1/2}} \sum_{d=0}^{p'-1} \binom{l-1}{2d+l-2p'} \binom{2d+1-2p'}{d} \left(\frac{e}{2}\right)^{2d+l-2p'} \qquad (2.2.17)$$

where

$$p' = \begin{cases} p \text{ for } p \le l/2 \\ l-p \text{ for } p \ge l/2 \end{cases}$$

Hence, the disturbing function for a nonspherical planet is given by

$$R = \sum_{l=2}^{\infty} \sum_{m=0}^{l} V_{lm} \qquad (2.2.18a)$$

where $V_{lm} = \frac{\mu R_e^l}{a^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S_{lmpq}(\omega, M, \Omega, \theta)$ (2.2.18b)

and $S_{lmpq} = \begin{cases} C_{lm}\cos\phi + S_{lm}\sin\phi \ , \ l-m \text{ even} \\ -S_{lm}\cos\phi + C_{lm}\sin\phi \ , \ l-m \text{ odd} \end{cases}$ (2.2.18c)

where $\phi = (l-2p)\omega + (l-2p+q)M + m(\Omega - \theta)$ (2.2.18d)

Or, using the conversions of Appendix B, write V_{lm} as

$$V_{lm} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G'_{lpq}(e) S_{lmq}(h, k, \lambda_N) \tag{B.6a}$$

where

$$S_{lmq} = e^{|q|} S^*_{lmq} \tag{B.6b}$$

$$S^*_{lmq} = \frac{1}{e^{|q|}} \begin{cases} \cos\xi\, e^{|q|} \cos q\omega + \sin\xi\, e^{|q|} \sin q\omega \,,\ l-m \text{ even} \\ \sin\xi\, e^{|q|} \cos q\omega - \cos\xi\, e^{|q|} \sin q\omega \,,\ l-m \text{ odd} \end{cases} \tag{B.3}$$

$$G'_{lpq} = G_{lpq} / e^{|q|} \tag{B.4}$$

$$\phi^* = (l-2p)\omega + (l-2p+q)M + m(\Omega-\theta) - m\lambda_{lm} \tag{B.1c}$$

Calculating the derivatives of the geopotential yields

$$\frac{\partial V_{lm}}{\partial a} = -\frac{(l+1)\mu R_e^l}{a^{l+2}} J_{lm} \sum_{p=0}^{l} F_{lmp} G'_{lpq} S_{lpq} \tag{2.2.19a}$$

$$\frac{\partial V_{lm}}{\partial h} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp} \left\{ G'_{lpq} \frac{\partial S_{lmq}}{\partial h} + S_{lmq} \frac{h}{e} \frac{dG'}{de} \right\} \tag{2.2.19b}$$

$$\frac{\partial V_{lm}}{\partial i} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} \frac{\partial F_{lmp}}{\partial i} G'_{lpq} S_{lpq} \tag{2.2.19c}$$

$$\frac{\partial V_{lm}}{\partial k} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp} \left\{ G'_{lpq} \frac{\partial S_{lmq}}{\partial k} + S_{lmq} \frac{k}{e} \frac{dG'}{de} \right\} \tag{2.2.19d}$$

$$\frac{\partial V_{lm}}{\partial \Omega} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} m \sum_{p=0}^{l} \frac{\partial F_{lmp}}{\partial i} G'_{lpq} \frac{1}{m} \frac{\partial S_{lmq}}{\partial \Omega},\ m \neq 0 \tag{2.2.19e}$$

$$\frac{\partial V_{lm}}{\partial \Omega} = 0, m = 0 \tag{2.2.19f}$$

$$\frac{\partial V_{lm}}{\partial \lambda_N} = 0 \tag{2.2.19g}$$

These results (eqn.s(2.2.19)) may be inserted into Lagrange's Planetary Equations (eqn.s(2.1.6)) resulting in

$$\frac{da}{dt} = 0 \tag{2.2.20a}$$

$$\frac{dh}{dt} = \frac{\sqrt{1-e^2}}{na^2}\left\{\frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp}\left[G'_{lpq}\frac{\partial S_{lmq}}{\partial k} + S_{lmq}\frac{k}{e}\frac{dG'}{de}\right]\right\}$$

$$-\frac{k\cot i}{na^2\sqrt{1-e^2}}\left\{\frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} \frac{\partial F_{lmp}}{\partial i} G'_{lpq} S_{lmq}\right\} \tag{2.2.20b}$$

$$\frac{di}{dt} = \frac{\cot i}{na^2\sqrt{1-e^2}\sin i}\frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l}$$

$$\times\left[k\left\{F_{lmp}\left[G'_{lpq}\frac{\partial S_{lmq}}{\partial h} + S_{lmq}\frac{h}{e}\frac{dG'}{de}\right]\right\}\right.$$

$$\left.-h\left\{F_{lmp}\left[G'_{lpq}\frac{\partial S_{lmq}}{\partial k} + S_{lmq}\frac{k}{e}\frac{dG'}{de}\right]\right\}\right] \tag{2.2.20c}$$

$$\frac{dk}{dt} = \frac{\sqrt{1-e^2}}{na^2}\frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp}\left[G'_{lpq}\frac{\partial S_{lmq}}{\partial h} + S_{lmq}\frac{h}{e}\frac{dG'}{de}\right]$$

$$+\frac{h\cot i}{na^2\sqrt{1-e^2}}\frac{\mu R_e^l}{a^{l+1}}J_{lm}\sum_{p=0}^{l}\frac{\partial F_{lmp}}{\partial i}G'_{lpq}S_{lmq} \tag{2.2.20d}$$

$$\frac{d\Omega}{dt}=\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\mu R_e^l}{a^{l+1}}J_{lm}\sum_{p=0}^{l}\frac{\partial F_{lmp}}{\partial i}G'_{lpq}S_{lmq} \tag{2.2.20e}$$

$$\frac{d\lambda_N}{dt}=\frac{n}{S_0}-\frac{d\theta_g}{dt}+\frac{1}{na^2S_0}\Bigg[\sqrt{1-e^2}\beta'$$

$$\times\left(\frac{\mu R_e^l}{a^{l+1}}J_{lm}\sum_{p=0}^{l}\left\{hF_{lmp}\left(G'_{lpq}\frac{\partial S_{lmq}}{\partial h}+S_{lmq}\frac{h}{e}\frac{dG'}{de}\right)\right.\right.$$

$$\left.\left.+kF_{lmp}\left(G'_{lpq}\frac{\partial S_{lmq}}{\partial k}+S_{lmq}\frac{k}{e}\frac{dG'}{de}\right)\right\}\right)$$

$$-2a\frac{\partial R}{\partial a}+\frac{S_0-\cos i}{\sqrt{1-e^2}\sin i}\frac{\mu R_e^l}{a^{l+1}}J_{lm}\sum_{p=0}^{l}\frac{\partial F_{lmp}}{\partial i}G'_{lpq}S_{lmq}\Bigg] \tag{2.2.20f}$$

Solar Wind Model

The acceleration of a satellite due to direct solar radiation pressure is given by(8:12)

$$\ddot{\underline{r}}=\gamma p_s\frac{A'}{m}\left(\frac{r_{sun}}{r_s}\right)^2\hat{r}_s \tag{2.3.1}$$

where $\underline{r}$ = position vector of satellite relative to planet center

$\underline{r}_{sun}$ = distance of sun at 1 A.U.

$\underline{r}_s$ = position of sun relative to planet center

$\underline{r}_v = \underline{r} - \underline{r}_s$

$p_s =$ radiation pressure on a perfectly absorbing surface

$\gamma -$ black body radiation constant ($= 1$)

$m =$ mass of the satellite

$A' =$ satellite reference area

$\hat{r} =$ unit vector designation

At times Venus will be between the sun and the satellite with Venus blocking the solar radiation and, hence, the need to compute the region for this solar occultation $g(u)$ is necessary. To do this, a right handed coordinate system (S_1, S_2, S_3) will be employed whose origin is at the center of Venus. S_1 will point at the sun and S_3 will point "up" out of the plane of the ecliptic. With this frame of reference, the conditions for occultation are

$$S_1 < 0 \tag{2.3.2a}$$

and

$$S_2^2 + S_3^2 \leq R_p^2 \tag{2.3.2b}$$

where R_p = planet radius plus atmospheric altitude blockage. If

$$r^2 = S_1^2 + S_2^2 + S_3^2$$

then eqn(2.3.2b) may be rewritten as

$$r^2 - S_1^2 \leq R_p^2$$

or

$$S_1 \leq -\sqrt{r^2 - R_p^2} \qquad (2.3.3)$$

(i.e., S_1 must be more negative). Now eqn(2.3.3) may replace eqn.s(2.3.2) as the occultation condition.

Next, assume the right ascension (α) and declination (δ) are known in some reference frame. Then, the cartesian coordinates of the satellite are given by

$$x_1 = r(\cos\Omega\cos u - \cos i\sin\Omega\sin u) \qquad (2.3.4a)$$

$$x_2 = r(\sin\Omega\cos u + \cos i\cos\Omega\sin u) \qquad (2.3.4b)$$

$$x_3 = r\sin i\sin u \qquad (2.3.4c)$$

where $u = \omega + f$. In order to relate the S_1, S_2, S_3-coordinate system to the cartesian coordinates of the satellite, two rotations will be performed:

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} = \begin{pmatrix} \cos(-\delta) & 0 & -\sin(-\delta) \\ 0 & 1 & 0 \\ \sin(-\delta) & 0 & \cos(-\delta) \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

or

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix} = \begin{pmatrix} \cos\delta\cos\alpha & \cos\delta\sin\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha & 0 \\ -\sin\delta\cos\alpha & -\sin\delta\sin\alpha & \cos\delta \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \qquad (2.3.5)$$

Using eqn(2.3.5), eqn(2.3.3) can be written as

$$S_1 = (\cos\delta\cos\alpha)x_1 + (\cos\delta\sin\alpha)x_2 + (\sin\delta)x_3 \leq -\sqrt{r^2 - R_p^2} \qquad (2.3.6)$$

For ease of manipulation, define

$$A = \cos\delta\sin\alpha \qquad (2.3.7a)$$

$$B = \sin\delta \qquad (2.3.7b)$$

$$C = \cos\delta\cos\alpha \qquad (2.3.7c)$$

Using eqn.s(2.3.4) and eqn.s(2.3.7), eqn(2.3.6) becomes

$$\begin{aligned} S_1 = \; & rC(\cos\Omega\cos u - \cos i\sin\Omega\sin u) \\ & + rA(\sin\Omega\cos u + \cos i\cos\Omega\sin u) \\ & + rB(\sin i\sin u) \\ & \leq -\sqrt{r^2 - R_p^2} \end{aligned} \qquad (2.3.8)$$

Again for simplicity, define

$$F = C\cos\Omega + A\sin\Omega \qquad (2.3.9a)$$

$$G = \cos i(A\cos\Omega - C\sin\Omega) + B\sin i \qquad (2.3.9b)$$

$$H = \sqrt{1 - \left(\frac{R_p}{r}\right)^2} \tag{2.3.9c}$$

Using eqn.s(2.3.9), the occultation condition is

$$g(u) = F\cos u + G\sin u + H(u) \leq 0 \tag{2.3.10}$$

There are three restrictions to the above occultation condition, $g(u)$. First, $g(u)$ depends on $\cos u$ and $\sin u$ causing it to be periodic with period 2π. Second, $g(u)$ also depends on $H(u)$ implying the minimum of $g(u)$ may not be zero. Finally, when $g(u)$ is at a minimum and $u=u_{min}$ then the satellite's orbit is occulted at least part of the time from the sun.

The following is necessary in order to calculate the orbital equations due to solar radiation. Since the orbital eccentricity of Venus is on the order of 0.0068, it can be assumed $r_s = a_p$ (where a_p = semi-major axis of the Venus orbit). Also assume $r_{sun} = a_e$ (a_e = semi-major axis of the earth orbit, e_e = 0.0167). Since the satellite is in orbit, assume $\underline{r}_v \approx -\underline{r}_s$ and the radiation pressure equation becomes

$$\ddot{\underline{r}} = -\frac{C_s A'}{m} f_s \tag{2.3.11}$$

$$\text{where } C_s = \gamma p_s \left(\frac{a_e}{a_p}\right)^2$$

Note that the above equation is of the form of a gradient of a potential (R):

$$\ddot{\underline{r}} = \nabla R^*$$

where $R^* = -\frac{C_s A'}{m} \underline{r} \cdot \hat{r}_s$

or $R^* = -\frac{C_s A'}{m} r \cos S$ (2.3.12)

Where S = planet centered angle between the sun and the satellite. The $\cos S$ may be determined (from eqn(2.3.8)) to be

$$\cos S = C(\cos\Omega + \cos u - \cos i \sin\Omega \sin u)$$

$$+ A(\sin\Omega \cos u + \cos i \cos\Omega \sin u)$$

$$+ B(\sin i \sin u)$$

The potential, R^*, is in the classical orbital set and has the following derivatives

$$\frac{\partial R^*}{\partial a} = -\frac{C_s A'}{m}\frac{r}{a}\cos S \qquad (2.3.13a)$$

$$\frac{\partial R^*}{\partial e} = -\frac{C_s A'}{m}\cos S \frac{\partial r}{\partial e} - \frac{C_s A'}{m} r \frac{\partial(\cos S)}{\partial u}\frac{\partial u}{\partial e}$$

$$\frac{\partial R^*}{\partial e} = -\frac{C_s A'}{m}\left[\cos S \frac{\partial r}{\partial e} + r \frac{\partial(\cos S)}{\partial u}\frac{\partial u}{\partial e}\right] \qquad (2.3.13b)$$

$$\frac{\partial R^*}{\partial i} = -\frac{C_s A'}{m} r \frac{\partial(\cos S)}{\partial i} \qquad (2.3.13c)$$

$$\frac{\partial R'}{\partial \Omega} = -\frac{C_s A'}{m} r \frac{\partial(\cos S)}{\partial \Omega} \qquad (2.3.13d)$$

$$\frac{\partial R'}{\partial \omega} = -\frac{C_s A'}{m} r \frac{\partial(\cos S)}{\partial u} \qquad (2.3.13e)$$

$$\frac{\partial R'}{\partial M} = \frac{\partial R}{\partial f}\frac{\partial f}{\partial M}$$

$$\frac{\partial R'}{\partial M} = -\frac{C_s A'}{m}\left(\frac{a}{r}\right)^2 \sqrt{1-e^2}\left[\cos S \frac{\partial r}{\partial u} + r\frac{\partial(\cos S)}{\partial u}\right] \qquad (2.3.13f)$$

In order to insert the above relationships into Lagrange's orbital equations, the following identities are needed:

$$\frac{\partial r}{\partial u} = \frac{r^2 e \sin f}{a(1-e^2)} \qquad (2.3.14a)$$

$$\frac{\partial r}{\partial e} = -a\cos f \qquad (2.3.14b)$$

$$\frac{\partial f}{\partial e} = \frac{\partial u}{\partial e} = \left(\frac{a}{r} + \frac{1}{1-e^2}\right)\sin f \qquad (2.3.14c)$$

$$\frac{\partial(\cos S)}{\partial i} = C(\sin i \sin \Omega \sin u)$$

$$+ A(-\sin i \cos \Omega \sin u) + B(\cos i \sin u) \qquad (2.3.14d)$$

$$\frac{\partial(\cos S)}{\partial \Omega} = C(-\cos \Omega \cos u - \cos i \cos \Omega \sin u)$$

$$+A(-\cos\Omega u-\cos i\sin\Omega\sin u) \qquad (2.3.14e)$$

$$\frac{\partial(\cos S)}{\partial u}=C(-\cos\Omega\sin u-\cos i\sin\Omega\cos u)$$

$$+A(-\sin\Omega\sin u+\cos i\cos\Omega\cos u)+B(\sin i\cos u) \qquad (2.3.14f)$$

In the above equations, the following partial derivatives have been used interchangeably(see eqn(2.4.19)):

$$\frac{\partial}{\partial f}=\frac{\partial}{\partial\omega}=\frac{\partial}{\partial u}$$

Lagrange's Planetary Equations, in the new variables, become

$$\frac{da}{dt}=\psi\frac{2}{na}\left(\frac{a}{r}\right)^2(1-e^2)\left(\cos S\frac{\partial r}{\partial u}+r\frac{\partial(\cos S)}{\partial u}\right) \qquad (2.3.15a)$$

$$\frac{dh}{dt}=\frac{\psi}{na^2\sqrt{1-e^2}}(\sin u(2+k\cos u+h\sin u)+h)r\frac{\partial(\cos S)}{\partial u}$$

$$-\frac{\psi\sqrt{1-e^2}}{na}\cos u\cos S-\frac{\psi k\cot i}{na^2\sqrt{1-e^2}}r\frac{\partial(\cos S)}{\partial i} \qquad (2.3.15b)$$

$$\frac{di}{dt}=\frac{\psi\cot i}{na^2\sqrt{1-e^2}}r\frac{\partial(\cos S)}{\partial u}-\frac{\psi}{na^2\sqrt{1-e^2}\sin i}\frac{\partial(\cos S)}{\partial\Omega} \qquad (2.3.15c)$$

$$\frac{dk}{dt}=\frac{\psi}{na^2\sqrt{1-e^2}}(\cos u(2+k\cos u+h\sin u)+k)r\frac{\partial(\cos S)}{\partial u}$$

$$+\frac{\psi\sqrt{1-e^2}}{na}\cos u\cos S-\frac{\psi h\cot i}{na^2\sqrt{1-e^2}}r\frac{\partial(\cos S)}{\partial i} \qquad (2.3.15d)$$

$$\frac{d\Omega}{dt}=\frac{\psi}{na^2\sqrt{1-e^2}}\sin i\, r\frac{\partial(\cos S)}{\partial i} \qquad (2.3.15e)$$

$$\frac{d\lambda_N}{dt}=\frac{\psi\beta'}{S_0na^2\sqrt{1-e^2}}(2+k\cos u+h\sin u)(k\sin u-h\cos u)r\frac{\partial(\cos S)}{\partial u}$$

$$-\frac{\psi}{S_0na^2}\left(a\beta'\sqrt{1-e^2}(k\cos u+h\sin u)+2r\right)\cos S$$

$$+\frac{\psi}{S_0na^2\sqrt{1-e^2}}\left(\frac{S_0-\cos i}{\sin i}\right)r\frac{\partial(\cos S)}{\partial i}+\frac{n}{S_0}-\frac{d\theta_g}{dt} \qquad (2.3.15f)$$

where $\psi=-\frac{C_r A'}{m}$

Atmospheric Drag Model

The force (D) per unit mass due to atmospheric drag is given by (21:81)

$$D=\frac{1}{2m}C_d\rho A v^2 \qquad (2.4.1)$$

where $D=$ drag

$m=$ mass

$C_d=$ drag coefficient

$\rho =$ atmospheric density $= \rho(r)$

$A =$ cross sectional area of the satellite

$v =$ satellite velocity

Using McCuskey's notation (19:81), the drag acts opposite to the velocity of the satellite and may, therefore, be written in vector notation as

$$\underline{D} = -D\hat{u}_T \tag{2.4.2}$$

where $\hat{u}_T$ is the unit vector along the tangent in the direction of motion.

For ease of calculation it is necessary to transform the drag in terms of the radial and transverse components of the satellite's flight path. The radial unit vector ($\hat{u}_r$) is along the position vector ($\underline{r}$) of the satellite orbit. The transverse unit vector ($\hat{u}_\theta$) is perpendicular to $\underline{r}$ in the orbital plane and forms an acute angle with the velocity vector ($\underline{v}$). In this coordinate system the drag can be written as

$$\underline{F} = R'\hat{u}_r + S'\hat{u}_\theta \tag{2.4.3}$$

In order to determine R' and S', the following relationships are employed:

$$R' = \underline{D} \cdot \hat{u}_r = -D(\hat{u}_T \cdot \hat{u}_r) \tag{2.4.4a}$$

$$S' = \underline{D} \cdot \hat{u}_\theta = -D(\hat{u}_T \cdot \hat{u}_\theta) \tag{2.4.4b}$$

To calculate the dot product of the above equations (eqn.s(2.5.4)) the velocity ($\underline{v}$) may be expressed as

$$\underline{v} = \dot{r}\hat{u}_r + r\dot{f}\hat{u}_\theta = v\hat{u}_T \tag{2.4.5a}$$

so

$$R' = D(\hat{u}_t \cdot \hat{u}_r) = -\frac{\dot{r}}{v}D \tag{2.4.5b}$$

$$S' = D(\hat{u}_t \cdot \hat{u}_\theta) = -\frac{r\dot{f}}{v}D \tag{2.4.5c}$$

The following two identities(19:148) will be needed:

$$r = \frac{a(1-e^2)}{1+e\cos f} \tag{2.4.6a}$$

$$r^2\dot{f} = h = \sqrt{k^2 M a(1-e^2)} \tag{2.4.6b}$$

Take the time derivative of eqn(2.4.6a)

$$\dot{r} = \frac{k\sqrt{M}\,e\sin f}{\sqrt{a(1-e^2)}} \tag{2.4.7a}$$

Also, note(19:148)

$$r\dot{f} = \frac{k\sqrt{M}(1+e\cos f)}{\sqrt{a(1-e^2)}} \tag{2.4.7b}$$

$$v = \frac{k\sqrt{M}(1+e^2+2e\cos f)^{1/2}}{\sqrt{a(1-e^2)}} \tag{2.4.7c}$$

Divide eqn(2.4.7a) by eqn(2.4.7c) and insert into eqn(2.4.5b) to get

$$R' = \frac{-e\sin f D}{\left(1 + e^2 + 2e\cos f\right)^{1/2}} \tag{2.4.8a}$$

Divide eqn(2.4.7b) by eqn(2.4.7c) and insert into eqn(2.4.5c) to get

$$S' = \frac{-(1 + e\cos f)D}{\left(1 + e^2 + 2e\cos f\right)^{1/2}} \tag{2.4.8b}$$

Now place R' and S' into the standard orbital reference frame with a unit vector ($\hat{P}$) along the perihelion and a unit vector ($\hat{Q}$) at an angle $f = 90^\circ$ to $\hat{P}$. This will make the drag

$$\underline{F} = \underline{E} = R'\hat{u}_r + S'\hat{u}_\theta$$

or

$$\underline{F} = (R'\cos f - S'\sin f)\hat{P} + (R'\sin f + S'\cos f)\hat{Q} \tag{2.4.9}$$

Note that, in this reference frame, the position is given by

$$\underline{r} = r\cos f\hat{P} + r\sin f\hat{Q}$$

or

$$\underline{r} = \frac{a\left(1 - e^2\right)\cos f}{(1 + e\cos f)}\hat{P} + \frac{a\left(1 - e^2\right)}{(1 + e\cos f)\sin f}\hat{Q} \tag{2.4.10}$$

Finally, calculate the partial derivatives of the perturbation (due to atmospheric drag) with respect to the orbital elements. Note

$$\frac{\partial R}{\partial c_j} = \frac{\partial R}{\partial r_i}\frac{\partial r_i}{\partial c_j}$$

or

$$\frac{\partial R}{\partial c_j} = \underline{F} \cdot \frac{\partial \underline{r}}{\partial c_j} \tag{2.4.11}$$

Where the c_j are the six classical orbital parameters.

Calculating the partial derivatives of the perturbation:

$$\frac{\partial R}{\partial a} = R'\frac{r}{a} \tag{2.4.12a}$$

$$\frac{\partial R}{\partial e} = -R'a\cos f + S'a\sin f\left(1 + \frac{r}{a(1-e^2)}\right) \tag{2.4.12b}$$

$$\frac{\partial R}{\partial M} = \frac{R'ea\sin f}{\sqrt{1-e^2}} + \frac{S'a^2\sqrt{1-e^2}}{r} \tag{2.4.12c}$$

$$\frac{\partial R}{\partial \Omega} = S'r\cos i \tag{2.4.12d}$$

$$\frac{\partial R}{\partial \omega} = S'r \tag{2.4.12e}$$

$$\frac{\partial R}{\partial i} = 0 \tag{2.4.12f}$$

Insert eqn.s(2.4.12) into Lagrange's Planetary Equations (eqn(2.1.1)) to obtain

$$\frac{da}{dt} = -\frac{C_d A}{m}\rho n a^2\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{3/2} \quad (2.4.13a)$$

$$\frac{de}{dt} = -\frac{C_d A}{m}\rho n a(e+\cos f)\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2} \quad (2.4.13b)$$

$$\frac{d\omega}{dt} = -\frac{C_d A}{m}\rho n a\frac{\sin f}{e}\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2} \quad (2.4.13c)$$

$$\frac{di}{dt} = 0 \quad (2.4.13d)$$

$$\frac{d\Omega}{dt} = 0 \quad (2.4.13e)$$

$$\frac{dM}{dt} = -\frac{C_d A}{m}\rho n a\frac{\sin f}{e}\frac{(1+e^2+e\cos f)}{(1+e\cos f)}\sqrt{1+e^2 2e\cos f} \quad (2.4.13f)$$

To transform the above equations into the orbital set $(a,h,i,k,\Omega,\lambda_M)$ recall (from eqn.s(2.1.5))

$$\frac{dh}{dt} = \frac{h}{e}\frac{de}{dt} + k\frac{d\omega}{dt} \quad (2.4.14a)$$

$$\frac{dk}{dt} = \frac{k}{e}\frac{de}{dt} - h\frac{d\omega}{dt} \quad (2.4.14b)$$

$$\frac{d\lambda_N}{dt}=\frac{1}{S_0}\left(\frac{dM}{dt}+\frac{d\omega}{dt}\right)+\frac{d\Omega}{dt}-\frac{d\theta_g}{dt} \qquad (2.4.14c)$$

Substituting the classical equations for atmospheric drag (eqn.s(2.4.13)) into the above equations yields

$$\frac{dh}{dt}=\frac{h}{e}\left[-\frac{C_d A}{m}\rho na(e+\cos f)\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2}\right]$$

$$+k\left[-\frac{C_d A}{m}\rho na\frac{\sin f}{e}\left(\frac{1+e^2 2e\cos f}{1-e^2}\right)^{1/2}\right]$$

$$\frac{dh}{dt}=-\frac{C_d A}{m}\rho na(h+\sin u)\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2} \qquad (2.4.15a)$$

$$\frac{dk}{dt}=\frac{k}{e}\left[-\frac{C_d A}{m}\rho na(e+\cos f)\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2}\right]$$

$$-h\left[-\frac{C_d A}{m}\rho na\frac{\sin f}{e}\left(\frac{1+e^2 2e\cos f}{1-e^2}\right)^{1/2}\right]$$

$$\frac{dk}{dt}=-\frac{C_d A}{m}\rho na(k+\cos u)\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2} \qquad (2.4.15b)$$

$$\frac{d\lambda_N}{dt}=\frac{1}{S_0}\left(-\frac{C_d A}{m}\rho na\frac{\sin f}{e}\frac{(1+e^2+e\cos f)}{(1+e\cos f)}\sqrt{1+e^2\cos f}\right.$$

$$\left.-\frac{C_d A}{m}\rho na\frac{\sin f}{e}\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)^{1/2}\right)$$

$$\frac{d\lambda_N}{dt} = -\frac{1}{S_0}\frac{C_d A}{m}\rho n a e \sin f\left[\beta' - \frac{r}{a\sqrt{1-e^2}}\right]\sqrt{1+e^2 2e\cos f} \quad (2.4.15c)$$

where $e\cos f = k\cos u + h\sin u$

$e\sin f = k\sin u - h\cos u$

$u = \omega + f$

Now, employ the Method of Averaging to eliminate the fast variables and retain only the averaged effects of atmospheric drag over one period. Introduce the averaged rate, given by (17:11)

$$\frac{\overline{d\zeta}}{dt} = \frac{1}{p}\int^{p}\frac{d\zeta}{dt}dt \quad (2.4.16)$$

where ζ = a, h, k, or λ_N

We can also relate time to the true anomaly (f) by

$$dt = \frac{r^2 df}{na^2\sqrt{1-e^2}} \quad (2.4.17)$$

so that

$$\frac{\overline{d\zeta}}{dt} = \frac{n}{2\pi}\int_0^{2\pi}\frac{d\zeta}{dt}\frac{r^2 df}{na^2\sqrt{1-e^2}} \quad (2.4.18)$$

The argument of periapsis (ω) will have only a small change over one period (see section 2.5), so write

$$df = d\omega + du \quad (2.4.19a)$$

or

$$df = du \tag{2.4.19b}$$

Substituting the orbital equations (eqn.(2.4.13.a) and eqn.s(2.4.15)) into the above integral (eqn(2.4.18)) yields

$$\frac{\overline{da}}{dt} = \frac{\gamma}{1-e^2}\int_0^{2\pi} \rho\left(\frac{r}{a}\right)^2 \left[1 + e^2 + 2(k\cos u + h\sin u)\right]^{3/2} du \tag{2.4.20a}$$

$$\frac{\overline{dh}}{dt} = \gamma\int_0^{2\pi} \rho\left(\frac{r}{a}\right)^2 (h + \sin u)$$

$$\times\left[1 + e^2 + 2(k\cos u + h\sin u)\right]^{1/2} du \tag{2.4.20b}$$

$$\frac{\overline{di}}{dt} = 0 \tag{2.4.20c}$$

$$\frac{\overline{dk}}{dt} = \gamma\int_0^{2\pi} \rho\left(\frac{r}{a}\right)^2 (k + \cos u)$$

$$\times\left[1 + e^2 + 2(k\cos u + h\sin u)\right]^{1/2} du \tag{2.4.20d}$$

$$\frac{\overline{d\Omega}}{dt} = 0 \tag{2.4.20e}$$

$$\frac{\overline{d\lambda_N}}{dt} = \gamma\int_0^{2\pi} \rho\left(\frac{r}{a}\right)^2 (k\sin u - h\cos u)\left[1 + e^2 + 2(k\cos u + h\sin u)\right]^{1/2}$$

$$\times\left(\beta' - \frac{r}{a\sqrt{1-e^2}}\right) du \tag{2.4.20f}$$

where $y = -\frac{C_d A}{m}\frac{na^2}{1-e^2}\frac{1}{2\pi}$

$\rho = \rho_0 e^{(h_0-h)/H}$

ρ_0 = reference density

h_0 = reference altitude

h = satellite altitude

H = scale height

<u>Resonance Model</u>

Whenever a parameter behaves in a sinusoidal manner and is effected by a sinusoidal forcing function the problem of resonance must be addressed. Resonance is a phenomenon whereby a displaced parameter approaches infinity when the frequency of the applied force equals the natural frequency of the parameter (9:47). Resonance effects are potentially present in the geopotential model (13:49) and will be described in this section. This section will also examine the elimination of the fast variables from the geopotential.

The geopotential may be written as

$$V_{lm} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S^*_{lmpq} \qquad (B.1a)$$

where

$$S^{*}_{lmpq}(\omega, M, \Omega, \theta, \lambda_N) = \begin{cases} \cos\phi^{*}, l-m \text{ even} \\ \sin\phi^{*}, l-m \text{ odd} \end{cases} \tag{B.1b}$$

$$\phi^{*} = (l-2p)\omega + (l-2p+q)M + m(\Omega - \theta) - m\lambda_{lm} \tag{B.1c}$$

Lagrange's Planetary Equations are concerned with the time rate of change of the orbital elements. The geopotential is a forcing function whose trigonometric argument is given by eqn(*B*.1*c*), above.

Fortunately, the rotational rate of Venus is retrograde and very slow with a value of -1.71460706e-05°/sec or 243 days per revolution. This means the geopotential will have a frequency opposite to the orbital elements and will be of no concern.

There are three orbital elements in eqn(*B*.1*c*) which can change with time: M, ω, Ω. The mean anomaly M varies with the speed of the orbital period through the mean motion. A typical period for a satellite about Venus is

$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$P = 1.6 \text{ hours} \tag{2.5.1}$$

where

$$a = 6387\,km$$

$$\mu = 3.257 \times 10^5\,km^3/s^2$$

The mean anomaly is a fast variable with a short period (relative to the five years of interest).

The argument of periapsis ω is considered a long period term and the longitude of the ascending node Ω is considered to be secular (smoothly varying with time) term (7:171).

Since the fast variable M is just an oscillation about the secular motion of the orbital elements it would be advantageous to average out its effects on the overall rate of change of the orbital elements. This can be accomplished by setting

$$(l - 2p + q) = 0$$

from eqn($B.2c$) so

$$q = 2p - l$$

Satellite Model

The satellite model is concerned with three parameters: a) the mass of the satellite m (used in the LPE, the solar wind model, and the atmospheric drag model), b) the drag coefficient of the satellite C_d (used in the atmospheric drag model), and c) the projected area of the satellite A' (used in the solar radiation model and the atmospheric drag model).

The mass of the satellite consists of the mass of several satellite subsystems. These subsystems are the Attitude and Control System

(AOCS), the Telemetry, Tracking, and Command (TT&C), the power system, the communication subsystems, the satellite antennas, and the satellite structure. Typical values of these subsystems will place a nominal value for the satellite mass at (20:67)

$$m = 1085\,kg \tag{2.6.1}$$

The next parameter of the satellite model is the drag coefficient C_d. When working with the drag coefficient of a satellite, several assumptions must be made concerning atmospheric molecules (14:14-15):

1) The satellite is considered to be stationary with the atmospheric molecules flowing past.

2) The molecules are assumed to impinge on the satellite, be retained temporarily on its surface, and then re-emitted.

3) The collisions between incident and re-emitted molecules are neglected.

There are several factors to consider in calculating C_d: the flow regime through which the satellite moves (reflected in the Knudsen number), the mechanism of molecular reflection (presented in the accommodation coefficient), and the satellite's dynamics and orientation to the atmospheric flow.

The type of flow is determined by the Knudsen number *Kn*, defined as the ratio of the mean free path of atmospheric molecules to the characteristic linear dimension of the satellite (11:184)

$$Kn = \frac{\lambda}{l} \tag{2.6.2}$$

Assuming a Maxwell distribution of particle velocities, the mean free path is

$$\lambda = \frac{1}{\sqrt{2}} \frac{1}{n\sigma}$$

where σ is the collision cross section of a molecule, given by

$$\sigma = 4\pi r^2$$

n is the molecular number given by

$$n = \frac{\rho}{m}$$

with a simplified density model for the atmosphere given by

$$\rho = \rho_0 e^{-aH}$$

and r is the collision radius of the molecule. The Knudsen number (eqn(2.6.2)) may now be written as

$$Kn = \frac{me^{aH}}{\sqrt{2}(4\pi r^2)\rho_0 l} \tag{2.6.3}$$

The atmosphere of Venus is 96.5% carbon dioxide (6:173) which has a collision radius of (18:A.8)

$$r = 2.0 x 10^{-12} km$$

Other typical values for the atmosphere of Venus are (18:157)

$$m = 7.31 \times 10^{-29} kg$$

$$\alpha = 4 \times 10^{-3} km^{-1}$$

$$H = 110 km$$

$$\rho_0 = 3.19 \times 10^{-4} kg/km$$

and assume a typical characteristic length for a satellite of

$$l = 5 \times 10^{-3} km$$

These values yield a Knutsen number of 10. In order to have free molecular flow, a Knudsen number of 10 or greater is needed (11:184). This implies an altitude greater than 110km may be considered free molecular flow. When a satellite is in this flow regime its drag coefficient is dependent on the molecular speed ratio. This is the ratio of the satellite speed to the probable molecular speed.

The next factor to consider is the mechanism of molecular reflection. The energy exchange between the satellite and the molecules depends on the speed and direction of the reflected molecules. It is assumed that the atmospheric molecules that impinge on the satellite's surface do not reflect specularly but attach themselves to the outer layer of the surface of the satellite for a period of time before being re-emitted. When the molecules disassociate from the satellite they are emitted

diffusively, having lost reference to their original direction of motion. This diffuse reflection is strongly dependent upon the nature of the satellite's surface and its structure (4:931).

The speed of the re-emitted molecules is a function of their kinetic temperature. During the period the molecules are attached to the satellite the molecules transfer some of their original temperature to the satellite. The amount of uncertainty (14:15) in the new temperature of the molecule is represented by the accommodation coefficient α, defined as

$$\alpha = \frac{T_i - T_r}{T_i - T_s}$$

where

T_i = original molecular temperature

T_r = re-emitted molecular temperature

T_s = satellite surface temperature

King-Hele suggests (14:15) that the accommodation coefficient is nearly one; therefore, a value of α = 1 is used for the satellite model. This amounts to elastic collisions.

The last factor to consider for the drag coefficient is the satellite's dynamics and its orientation to the molecular flow. For a cylinder tumbling end over end the drag coefficient can be related to all the parameters, above, by (4:940)

$$C_d = 2\left\{1 + \frac{\pi^2(l+d)}{6(4l+d)}(1-\alpha)^{1/2}\right\} \tag{2.6.4}$$

where

l = cylinder length

d = cylinder diameter

With α = 1, the drag coefficient becomes

$$C_d = 2 \tag{2.6.5}$$

The last parameter of interest for the satellite model is the projected area. This area affects the atmospheric drag of the satellite and the pressure of solar radiation on the satellite. The size of the area depends on the orientation of the satellite as it orbits Venus. At one extreme, the area can be taken to be the projected area of the side of a cylinder so that $A=l+d$. At the other extreme, the area is the end of the cylinder, or $A=\pi d^2/4$.

It would be unrealistic to assume a satellite with no station keeping abilities would remain in any particular orientation with respect to the atmosphere or the solar radiation. This precludes the use of either extreme of reference area mentioned above. An uncontrolled satellite, with any initial rotational motion, would begin to spin about its axis of greatest moment of inertia. This is due to relatively small external torques. As the satellite tumbles about its orbit it can assume any

orientation with respect to the planet's atmosphere and the solar radiation. A mean value of the projected area for a tumbling satellite is given by King-Hele to be

$$A' = ld(0.818 + 0.25d/l) \tag{2.6.6}$$

Typical values for length and diameter of a satellite are

l = 11.60

d = 2.38

These values give a projected area for the satellite of

$$A' = 24m^2 \tag{2.6.7}$$

To summarize, the parameters for the satellite model are

$$m = 1085kg \tag{2.6.8a}$$

$$C_d = 2 \tag{2.6.8b}$$

$$A' = 24m^2 \tag{2.6.8c}$$

III. Approach and Results

Several simplifying assumptions were made in order to model the hypersurface. The first assumption was with respect to the geopotential model. The other assumptions were made on the values used for the orbital elements.

For the geopotential, a 4x4 gravity field was considered. The coefficients for the gravity field are give in Table 3.2.

Restrictions on the orbital parameters consisted of two types. The first type of restriction was to choose an initial value for a parameter and never vary it. This was done for two parameters, the mean anomaly (M) and the argument of periapsis (ω). The mean anomaly is a fast variable and has been averaged out in the modeling of the previous chapter. For this reason, any initial value of M is arbitrary. Therefore, a value of zero has been chosen. In order to bring the hypersurface down to a conceptually understandable four dimensions, another constant orbital parameter was needed. The argument of periapsis was chosen with a value of 53° so calculations of the time-to-impact on the planet could be started with the periapsis of the satellite's orbit in eclipse whenever the longitude of the ascending node of the satellite is zero. The 53° relates to the longitude of the ascending node of Venus for the epoch used in this study. This is shown in Fig(3.1), below.

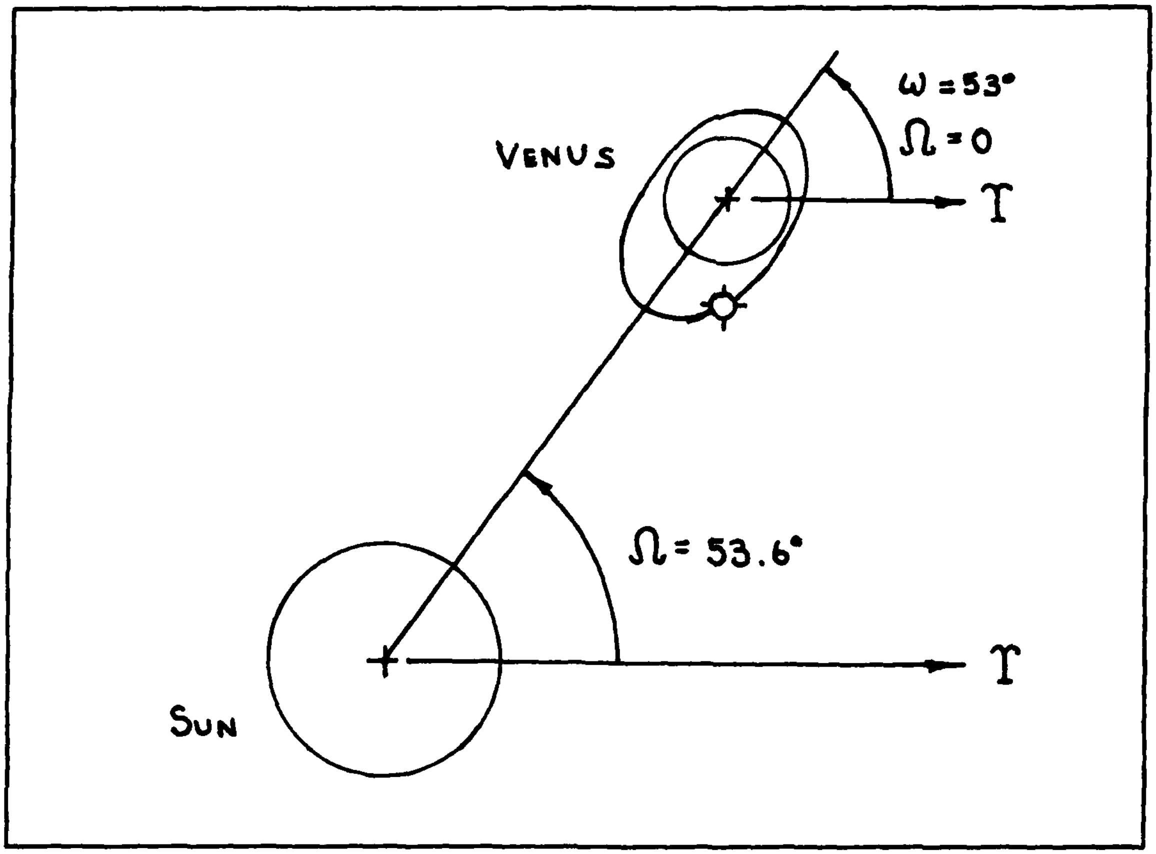

Figure 3.1: Orientation of the Satellite to Venus and the Sun

The second type of restriction was in the range of the parameters which were allowed to vary. This restriction was placed on two of the orbital parameters, the eccentricity (e) and the inclination (i). The eccentricity was kept small ($0.02 \leq e \leq 0.10$) and the inclination close to the equator ($0.5 \leq i \leq 15.5$).

This leaves two orbital parameters: the longitude of the ascending node and the semi-major axis. The longitude of the ascending node (Ω) was varied throughout its entire range from zero to 2π. The semi-major axis was related to the perigee altitude by

$$h_p = a\left(1 - e^2\right) - r \tag{3.1}$$

and treated as the dependent variable.

Now there is a four dimensional surface to use as a *window of survivability*. By plotting the relationship of

$$h_p = F(i, e, \Omega) \tag{3.2}$$

it can be seen, given a satellite's orbital parameters, how low in altitude a satellite can go and still have five years before impact.

Given the constraints listed above, the models of the perturbations were programmed using the FORTRAN programming language incorporating programs from JPL (16). The computer employed was an IBM XT compatible computer, specifically, a Connextion XT Turbo with an Intel 8088-2 microprocessor and an Intel 8087 math coprocessor.

Once the program for integrating the LPE was constructed a criterion for the satellite's entry into the atmosphere was needed. In other words, how low in altitude must the satellite be before it is considered to be within the atmosphere of Venus? Recall from Section 2.6, the interaction of the satellite with the atmosphere of Venus can be considered as random molecular collisions for altitudes above 110km. This

is due to a Knudsen number $\geq$ 10. For this reason, the satellite was considered to be within the atmosphere when it went below 110km in altitude. A sample orbital calculation using the programmed model shows the rapid orbital deterioration when the periapsis altitude falls below 110km. See Fig(3.2), below.

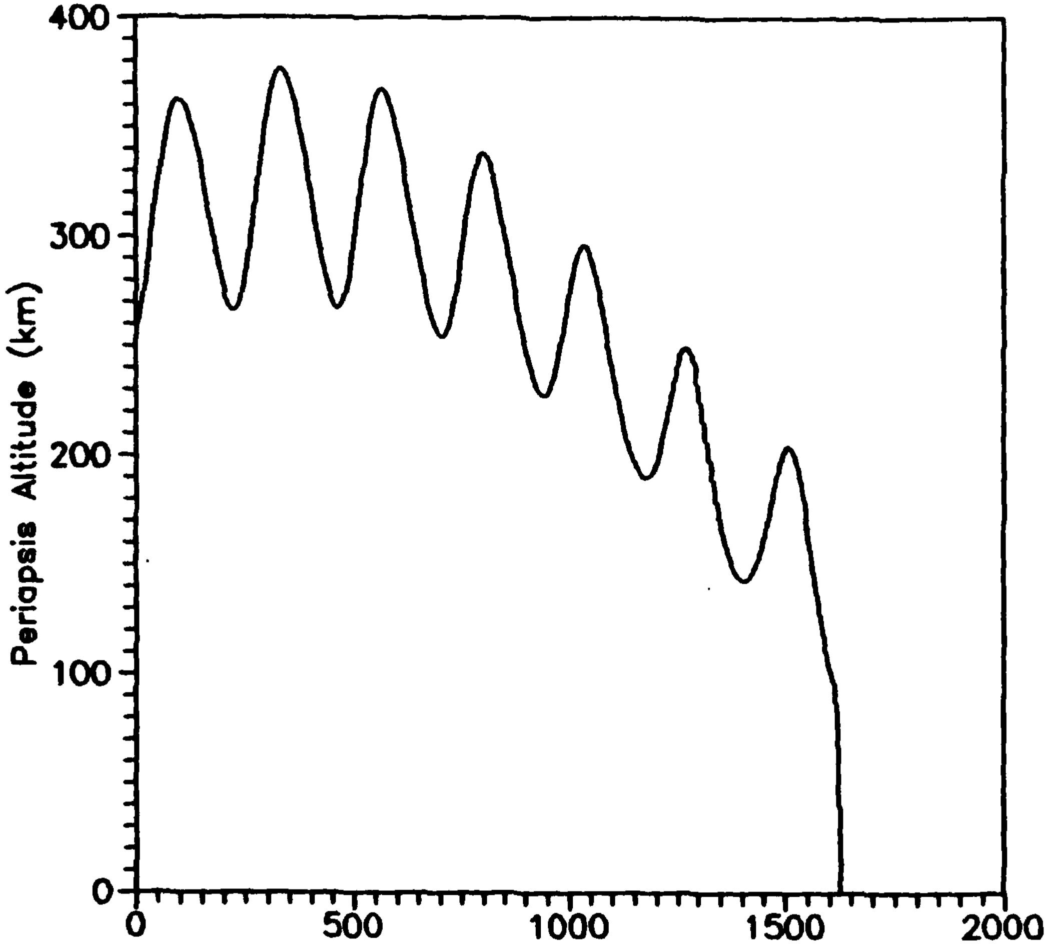

Figure 3.2 Periapsis Altitude

In order to account for any inaccuracies in the model a margin of safety of 60 days was added. The satellite was considered to have a five year survivability if it went below a 110 km altitude after 5 years and 60 days (1885 days).

There were 720 distinct points in the four dimensional space examined. Each computer integration run of the LPE took approximately 30 minutes. It took roughly four or five runs at each point to find the initial periapsis altitude needed to insure a five year survivability of the satellite.

The epoch of the initial conditions was 23 February 1990. The data for Venus on this date are listed in the table 3.1, below (2).

Table 3.1: Venus Data	
Parameter	Value
Epoch	19900223
$a(km)$	108208900.
e	.006777729
$i(°)$	2.6620179
$\Omega(°)$	53.555983
$\omega(°)$	-105.23265
$M(°)$	43.540609
$n(°/sec)$	1.854317699e-5
Prime Meridian	91.561209

The coefficients for the geopotential are listed in Table 3.2.

Table 3.2: Geopotential Coefficients			
n	m	C_{nm}	S_{nm}
2	0	-.4520722821771336e-05	.0000000000000000e+00
2	1	.2039874602320317e-06	.3981774131465276e+02
2	2	.6582161633245868e-06	-.6165295905454911e+01
3	0	.1342079593314583e-05	.0000000000000000e+00
3	1	.2496259200880112e-05	.1014747625063479e+02
3	2	.2690349661463819e-06	.424676198132208e+02
3	3	.3130468881843509e-07	.4022519369940560e+02
4	0	.2413459619768915e-05	.0000000000000000e+00
4	1	.7032376618616599e-06	.1321605322173613e+03
4	2	.1151665471340579e-06	.4338487639282226e+02
4	3	.1564751078966026e-07	-.4872890600202502e+02
4	4	.2868983692949129e-07	.2042893824377979e+02

Other parameters needed for orbital calculations are given in Table 3.3.

Table 3.3: Miscellaneous Parameters	
Parameter	Value
$\mu G(km^3/\sec^2)$	3.2485877e05
Venus Radius (km)	6051.0
Venus Rotation Rate(° /sec)	-1.71460706e-05
$\mu_{sun} G(km^3/\sec^2)$	0.13271244e12

The next step in the development of the hypersurface was to fit the data generated from the model to a least squares curve. The first step in deriving a least square representation in multidimensions was to define the dependent and independent variables

x_i = independent variable(s) (may be multidimensional)

y_i = measured dependent variable

For a least squares approximation the data is to be fitted to a multiple of functions

$$y = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_m f_m(x)$$

or

$$y = \sum_{i=1}^{m} c_i f_i(x) \tag{3.3}$$

Assume that there are n measurements of y, so eqn(3.3) may be expanded for each value of y:

$$y_1 = c_1 f_1(x_1) + c_2 f_2(x_1) \ldots c_m f_m(x_1)$$

$$y_2 = c_1 f_1(x_2) + c_2 f_2(x_2) \ldots c_m f_m(x_2)$$

$$\vdots$$

$$y_n = c_1 f_1(x_n) + c_2 f_2(x_n) \ldots c_m f_m(x_n)$$

Define the residuals as

$$R_1 = y_1 - \sum_{i=1}^{m} c_i f_i(x_1)$$

$$R_2 = y_2 - \sum_{i=1}^{m} c_i f_i(x_2)$$

$$\vdots$$

$$R_n = y_n - \sum_{i=1}^{m} c_i f_i(x_n)$$

In order to get the best least-squares-fit to the chosen functions, the sum of the squares of the residuals must be minimized. That is to say

$$R = \sum_{j=1}^{n} R_j^2 = \text{minimum} \tag{3.4}$$

Define the total derivative of the residual (for later use) as

$$dR = \frac{\partial R}{\partial c_1} dc_1 + \frac{\partial R}{\partial c_2} dc_2 + \ldots + \frac{\partial R}{\partial c_m} dc_m$$

To find the minimum, set the derivative of eqn(3.4) to zero.

$$\frac{\partial R}{\partial c_k} = 0 = \frac{\partial}{\partial c_k} \sum_{j=1}^{n} R_j^2 = 2 \sum_{j=1}^{n} R_j \frac{\partial R_j}{\partial c_k} \qquad k = 1, 2, 3, \ldots, m$$

but $\frac{\partial R_j}{\partial c_k} = f_k(x_j)$

so

$$\sum_{j=1}^{n} R_j f_k(x_j) = \sum_{j=1}^{n}\left[y_j - \sum_{i=1}^{m} c_i f_i(x_j)\right] f_k(x_j) = 0 \qquad (3.5)$$

Rearranging terms in eqn(3.5) yields

$$\sum_{j=1}^{n} y_j f_k(x_j) = \sum_{i=1}^{m} c_i \sum_{j=1}^{n} f_i(x_j) f_k(x_j) \qquad k = 1, 2, \ldots, m \qquad (3.6)$$

Next, put eqn(3.6) in matrix form for ease of manipulation. In order to accomplish this, some definitions need to be made. Define

$$b_k = \sum_{j=1}^{n} y_j f_k(x_j)$$

$$a_{ki} = \sum f_i(x_j) f_k(x_j)$$

so

$$b_k = \sum_{i=1}^{m} c_i a_{ki}$$

Now, define

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

$$B = \begin{Bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{Bmatrix}$$

$$C = \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{Bmatrix}$$

such that $B=AC$. Also, note

$$y = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{Bmatrix}$$

$$F = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_m(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_m(x_2) \\ \vdots & \vdots & & \vdots \\ f_1(x_n) & f_2(x_n) & \dots & f_m(x_n) \end{pmatrix}$$

so

$$A = F^T F$$

$$B = F^T y$$

Therefore, eqn(3.7) may now be written as

$$F^T y = (F^T F) C \tag{3.7}$$

Since y is given and F is chosen to fit the data, the interest is in finding a C that will minimize the residuals. This will give a best fit of the data for the chosen functions. The vector C is (from eqn(3.7))

$$C = (F^T F)^{-1} (F^T y) \tag{3.8}$$

It now becomes a matter of judicious selections of functions to best fit the calculated periapsis altitudes.

The calculated values for periapsis altitudes for inclinations of 0.5° and 1.5° are shown in Table 3.4 (at the end of this section) along with graphs of the results, Fig(3.1) and Fig(3.3), and contour maps of the data, Fig(3.2) and Fig(3.4). The remainder of the calculated data may be found in Appendix *D* along with graphs in Appendix *E*.

From analysis of the data the periapsis altitude (h_p) behaves as a sinusoidal function with the ascending node. The h_p behaves as a inverse fourth root in eccentricity and as a quadratic in inclination. See Fig(3.1) and Fig(3.3). For the trends in the longitude of the ascending node refer to Fig(E.33). The functions used to fit the calculated periapsis altitude are

$$f_1(i,e,\Omega) = e^{-1/4}$$

$$f_2(i,e,\Omega) = \sin(\Omega - 149°)$$

$$f_3(i,e,\Omega) = i(1-i)\pi/180°$$

These functions result in a least squares fit equation of

$$h_p = 100.6829e^{-1/4} + 48.7606\sin(\Omega - 149°) + 0.10607i(i+103.9)$$

where the inclination is in radians. The correlation coefficient for eqn(3.9) is

$$r = 0.97257 \qquad (3.10)$$

where the correlation coefficient is defined by (10:87)

$$r = \left[1 - \frac{\sigma^2_{hp.x}}{\sigma^2_{hp}}\right]^{1/2}$$

with

$$\sigma_{hp,x} = \left[\frac{\sum_{i=1}^{n} (h_{xi} - hp_{ic})^2}{n-2} \right]^{1/2}$$

$$\sigma_{hp} = \left[\frac{\sum_{i=1}^{n} (hp_i - hp_m)^2}{n-1} \right]^{1/2}$$

$$h_{pm} = \frac{1}{n} \sum_{i=1}^{n} hp_i$$

h_{pi} = calculated value of h_p from the model

h_{pic} = least squares calculated value of h_p

The least squares representation stays within $\pm$20km throughout most of the range of parameters examined. This is a 10% variation, well within engineering practice. The range of data that does not support this small variation is

$$10.5^\circ \leq i \leq 15.5^\circ$$

$$e = 0.02$$

$$-30^\circ \leq \Omega \leq 60^\circ$$

In this range, the error is as high as 50km, however, all the deviations from the computed model are on the positive side of the hypersurface and, therefore, conservative..

This region of high deviation constitutes less than 3% of the entire hypersurface. If it is excluded from consideration, then the least squares equation becomes

$$h_p = 100.73e^{-1/4} + 50.265\sin(\Omega - 149°) + 0.0872i(i+1)$$

with a correlation coefficient of

$$r = 0.98570$$

TABLE 3.4: INSERTION WINDOW HYPERSURFACE (i = 0.5 and 1.5)							
$i(°)$	e	$\Omega(°)$	Hp (Km)	$i(°)$	e	$\Omega(°)$	Hp (Km)
0.50	0.02	0	235.41	1.50	0.02	0	236.48
0.50	0.02	15	223.84	1.50	0.02	15	224.82
0.50	0.02	60	211.20	1.50	0.02	60	209.14
0.50	0.02	105	240.89	1.50	0.02	105	235.50
0.50	0.02	150	279.41	1.50	0.02	150	273.63
0.50	0.02	195	304.99	1.50	0.02	195	300.67
0.50	0.02	240	310.18	1.50	0.02	240	307.73
0.50	0.02	285	294.01	1.50	0.02	285	293.23
0.50	0.02	330	261.18	1.50	0.02	330	261.67
0.50	0.04	0	196.68	1.50	0.04	0	195.24
0.50	0.04	15	184.30	1.50	0.04	15	182.66
0.50	0.04	60	171.82	1.50	0.04	60	168.26
0.50	0.04	105	203.59	1.50	0.04	105	198.12
0.50	0.04	150	249.38	1.50	0.04	150	243.82
0.50	0.04	195	280.87	1.50	0.04	195	276.55
0.50	0.04	240	286.54	1.50	0.04	240	283.66
0.50	0.04	285	265.80	1.50	0.04	285	263.88
0.50	0.04	330	225.96	1.50	0.04	330	224.42
0.50	0.06	0	177.44	1.50	0.06	0	175.28
0.50	0.06	15	165.78	1.50	0.06	15	163.43
0.50	0.06	60	154.03	1.50	0.06	60	150.46
0.50	0.06	105	183.93	1.50	0.06	105	178.85
0.50	0.06	150	230.17	1.50	0.06	150	225.00
0.50	0.06	195	262.79	1.50	0.06	195	258.75
0.50	0.06	240	268.53	1.50	0.06	240	265.52
0.50	0.06	285	246.53	1.50	0.06	285	244.27
0.50	0.06	330	206.02	1.50	0.06	330	203.85
0.50	0.08	0	164.89	1.50	0.08	0	162.59
0.50	0.08	15	153.94	1.50	0.08	15	151.46
0.50	0.08	60	142.81	1.50	0.08	60	139.31
0.50	0.08	105	170.59	1.50	0.08	105	165.99
0.50	0.08	150	215.58	1.50	0.08	150	210.89
0.50	0.08	195	248.15	1.50	0.08	195	244.38
0.50	0.08	240	253.39	1.50	0.08	240	250.63
0.50	0.08	285	231.50	1.50	0.08	285	229.10
0.50	0.08	330	192.12	1.50	0.08	330	189.82
0.50	0.10	0	155.58	1.50	0.10	0	153.33
0.50	0.10	15	145.32	1.50	0.10	15	142.89
0.50	0.10	60	134.52	1.50	0.10	60	131.46
0.50	0.10	105	160.44	1.50	0.10	105	156.39
0.50	0.10	150	203.82	1.50	0.10	150	199.50
0.50	0.10	195	235.41	1.50	0.10	195	231.99
0.50	0.10	240	240.36	1.50	0.10	240	237.75
0.50	0.10	285	218.85	1.50	0.10	285	216.60
0.50	0.10	330	181.23	1.50	0.10	330	178.88

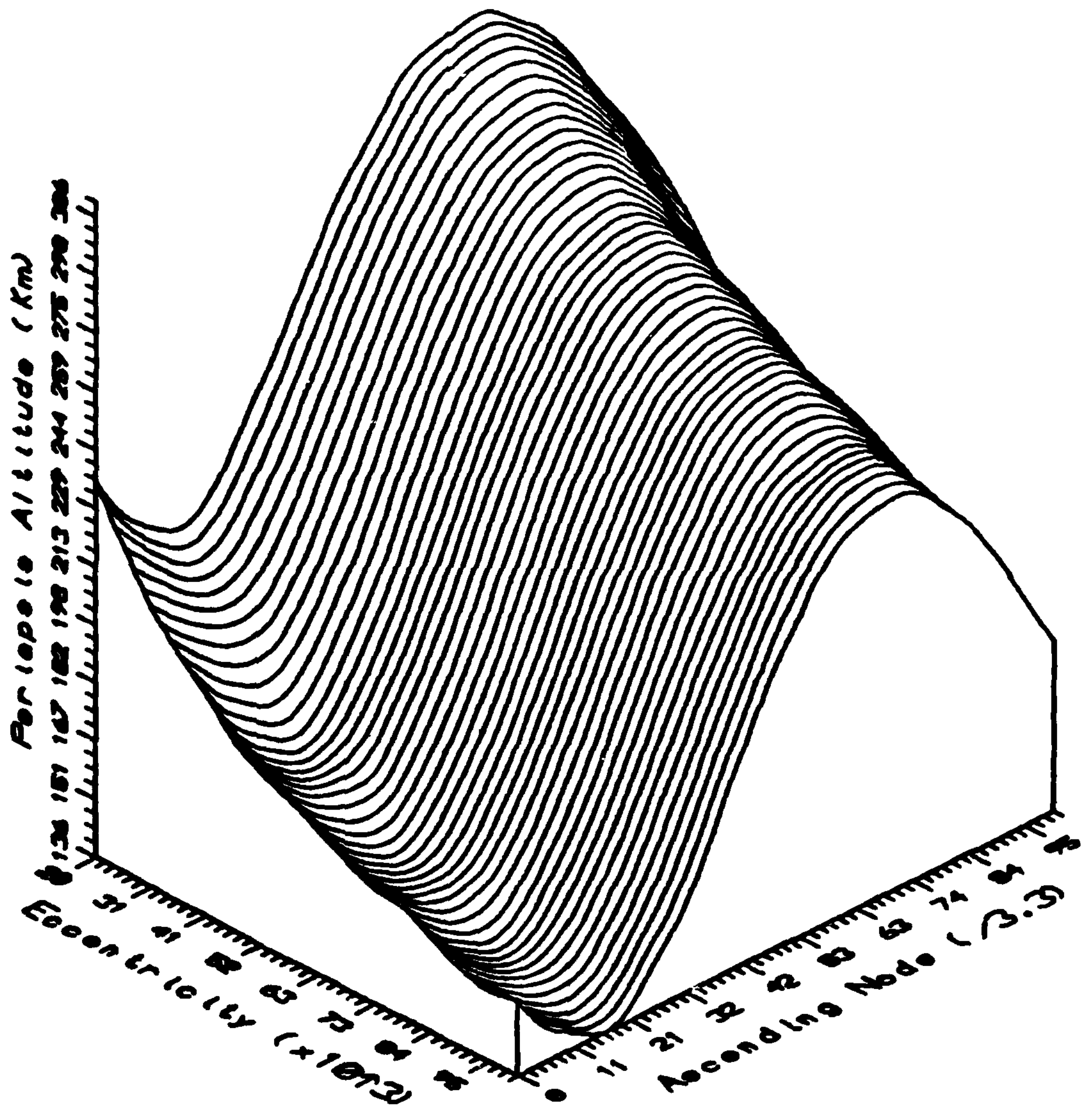

Fig 3.3: Periapsis Altitude (i = 0.5 deg)

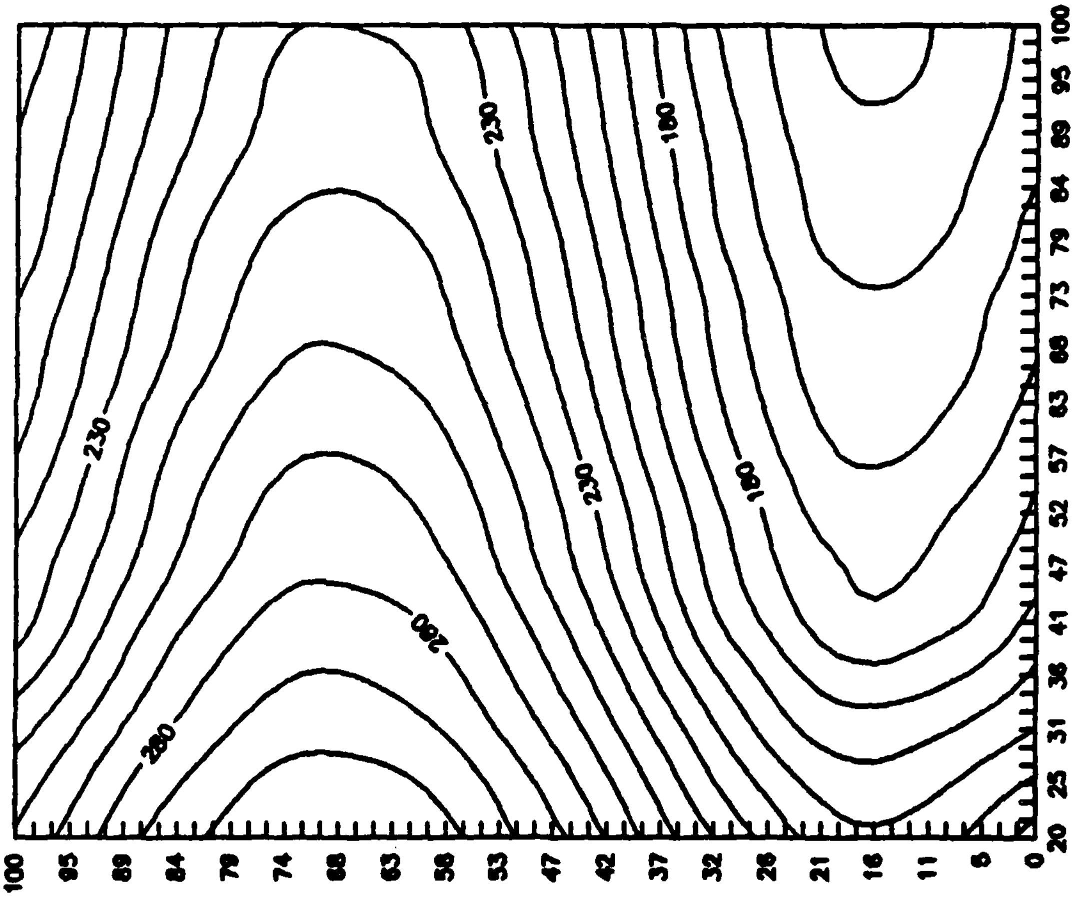

Fig 3.4: Periapsis Contour (i = 0.5 deg)

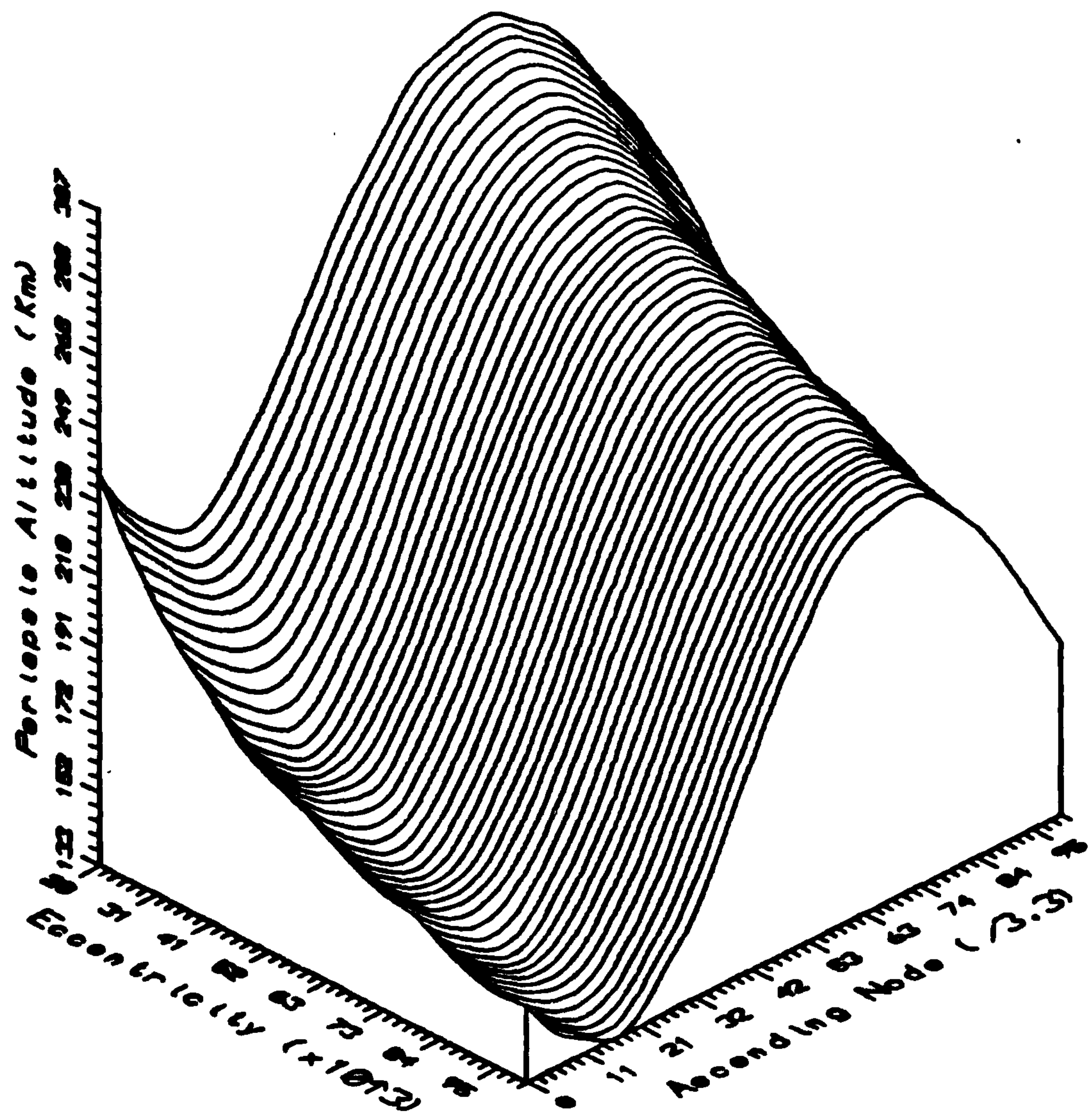

Fig 3.5: Periapsis Altitude (i = 1.5 deg)

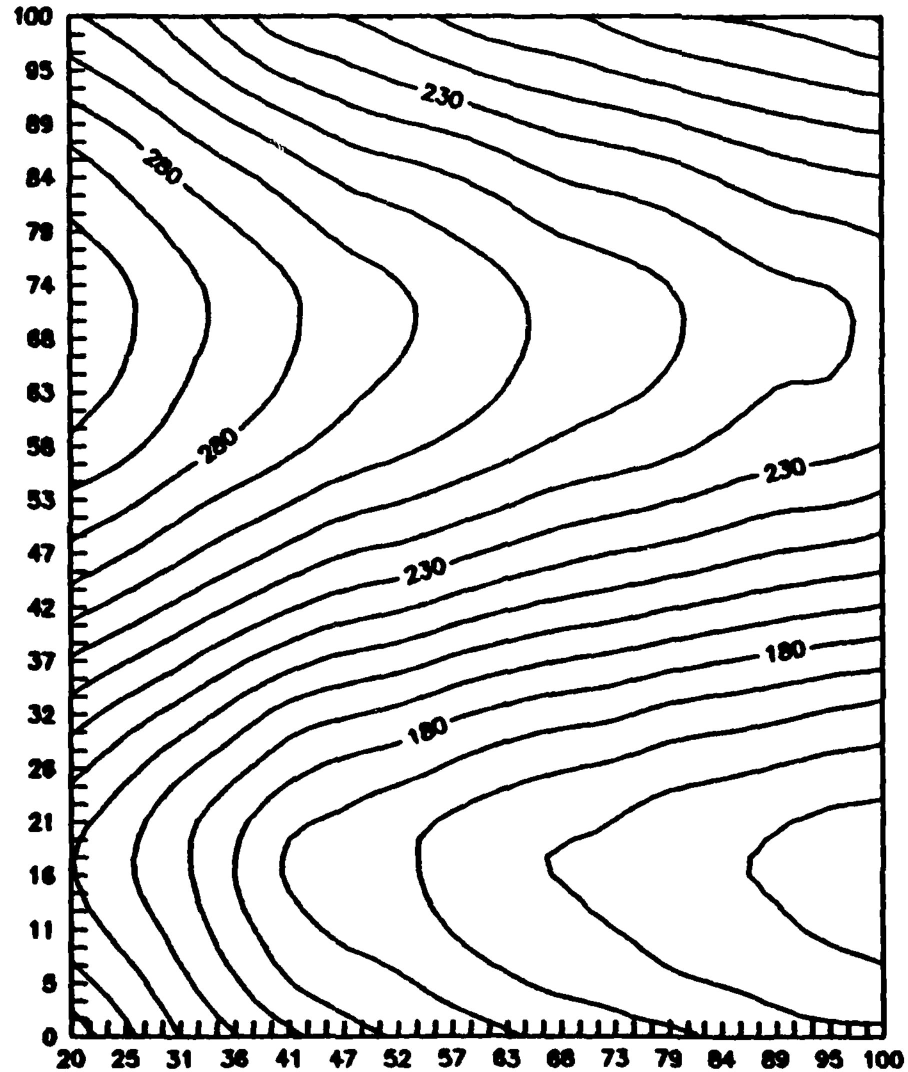

Fig 3.6: Periapsis Contour (i = 1.5 deg)

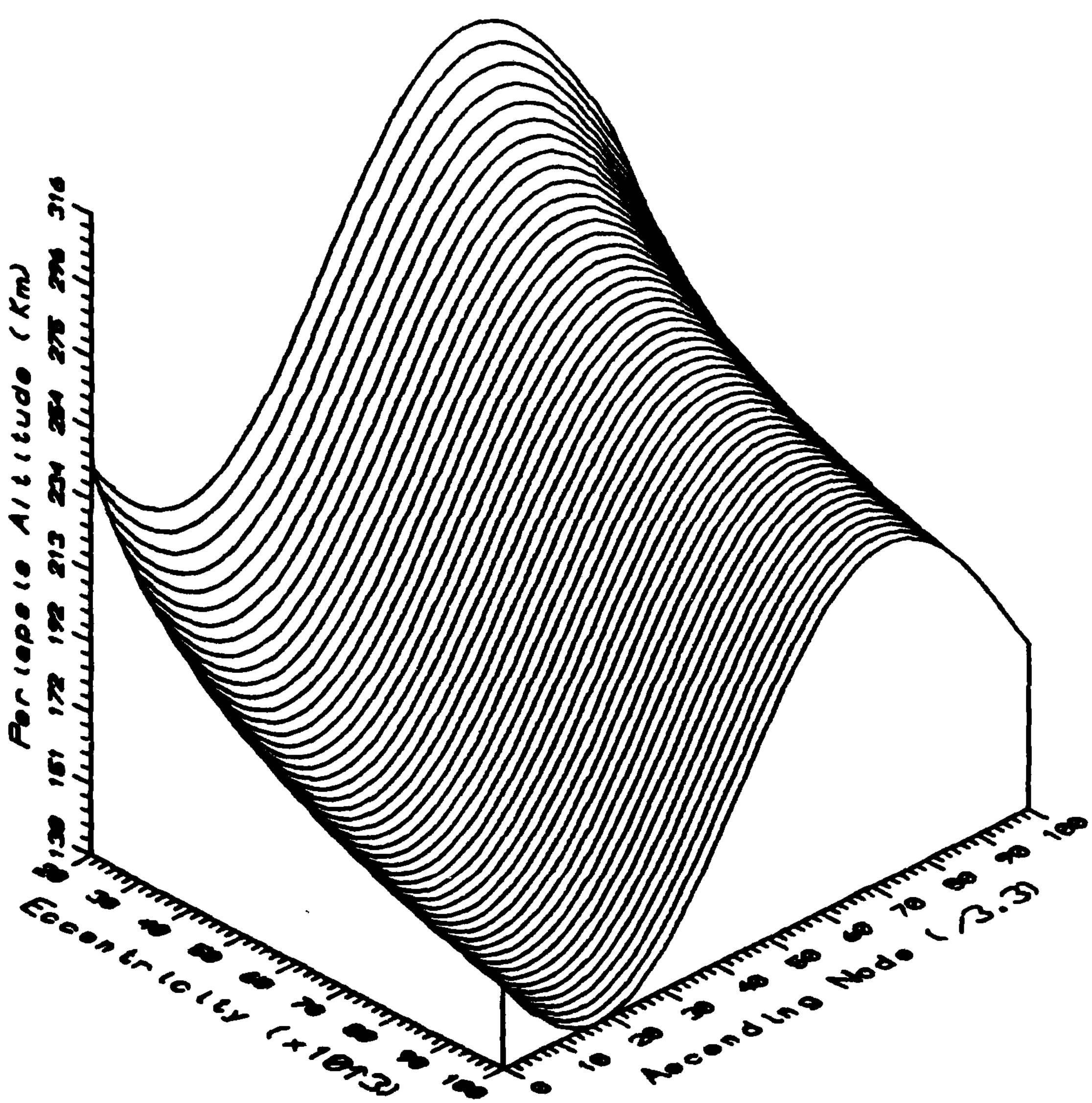

Fig 3.7: Periapsis Altitude (i = 0.5 deg)

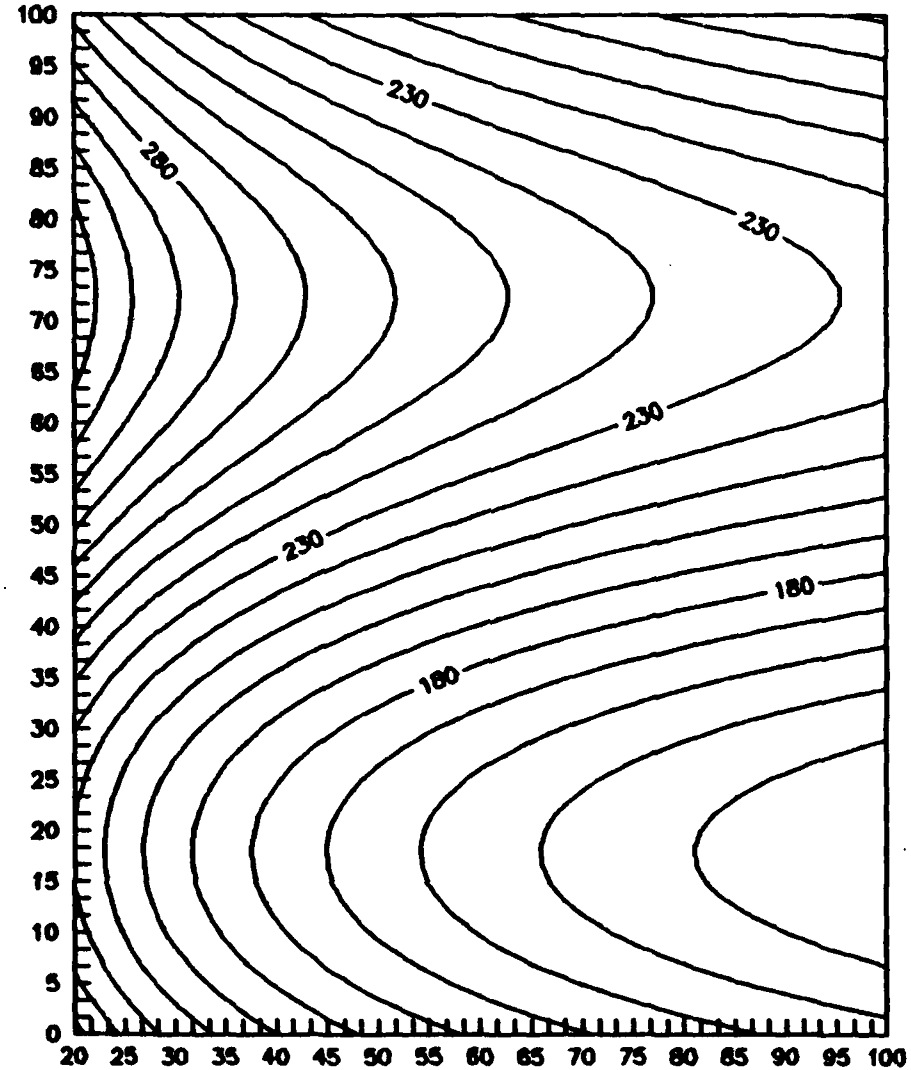

Fig E.8: Periapsis Contour (i = 0.5 deg)

IV. Conclusions and Recommendations

Conclusions

This effort investigated the creation of an analytic function in the six dimensional orbital parameters space that separated a satellite's orbit between a five year or greater survivability and a satellite's potential for entering a planet's atmosphere in under five years. On the basis of this investigation, the following conclusions are made:

1. It is possible to generate smooth curves in the orbital parameter space that separate a satellite's life expectancy as a function of periapsis altitude and time.

2. The periapsis altitude (h_p) behaves as a sinusoidal function with the ascending node. The h_p behaves as a inverse fourth root in eccentricity and a quadratic in inclination.

3. It is possible to generate a survivability function h_p for the hypersurface with an acceptable engineering error of 10%. The approximation generated in this thesis is

$$h_p = 100.6829e^{-1/4} + 48.7606\sin[(\Omega - 149^\circ)\pi/180] + 0.10607i(i+1)$$

with a correlation coefficient of

$$\sigma = 0.98570$$

where the constraints in this thesis are

$$M = 0°$$

$$\omega = 53°$$

$$0.02 \leq e \leq 0.10$$

$$0° < i < 15°$$

$$0° \leq \Omega \leq 360°$$

Recommendations

Based on the findings of this investigation, the following recommendations for further study are proposed:

1. Investigation of the behavior of the satellite for inclinations up to a value of 90° and eccentricities above .10 should be conducted.

2. Further study of the modeling of the hypersurface using *simple* functions (sin x, cos x, e^x, etc.)

3. A layered mapping of five year increments above the planet Venus should be performed so, should a satellite lose its station keeping abilities, a determination can be made at once concerning the lifetime of the satellite.

Appendix A

Conversion of the Geopotential into the Classical Orbital Elements

Although the following derivation has been outlined by Kaula (13:30-37) it is included here for clarification of the geopotential model in Section 2.2. This derivation of the geopotential deviates from that presented by Kaula in that Hansen's coefficients are used in formulating the Eccentricity Function.

$$V(r,\theta,\phi) = -\frac{\mu}{r}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{r}{R_e}\right)^{-l} P_l^m(\sin\theta)\left(C_{lm}\cos m\phi + S_{lm}\sin m\phi\right) \qquad (A.1)$$

The geopotential in its spherical harmonic representation (eqn(2.3.15)) must be translated into a Keplarian representation. To do this, two special functions will be introduced: the Inclination Function ($F_{lmp}(i)$) and the Eccentricity Function ($G_{lpq}(e)$), both to be defined later. In order to use these two functions the geopotential must first be rearranged.

The first step will be to get the geopotential into a form compatable with the Inclination Function. The following relationships will be needed.

$$\cos mx = Re\left\{e^{mjx}\right\} = Re\left\{(\cos x + j\sin x)^m\right\}$$

or

$$\cos mx = Re\left\{\sum_{s=0}^{m}\binom{m}{s} j^{s}\cos^{m-s}x\sin^{s}x\right\} \tag{A.2a}$$

In like manner, note

$$\sin mx = Re\left\{\sum_{s=0}^{m}\binom{m}{s} j^{s-1}\cos^{m-s}x\sin^{s}x\right\} \tag{A.2b}$$

Multiply the two equations, above, to get

$$\sin^{\alpha}x\cos^{\beta}x = \left(\frac{e^{jx}-e^{-jx}}{2j}\right)^{\alpha}\left(\frac{e^{jx}+e^{-jx}}{2}\right)^{\beta}$$

$$= \frac{(-1)^{\alpha}j^{\alpha}}{2^{\alpha}}\sum_{g=0}^{\alpha}(-1)^{g}\binom{\alpha}{g}e^{(\alpha-g)jx}e^{-gjx}$$

$$\times\frac{1}{2^{\beta}}\sum_{h=0}^{\beta}\binom{\beta}{h}e^{(\beta-h)jx}e^{-hjx}$$

$$\sin^{\alpha}x\cos^{\beta}x = \frac{(-1)^{\alpha}j^{\alpha}}{2^{\alpha+\beta}}\sum_{g=0}^{\alpha}\sum_{h=0}^{\beta}(-1)^{g}\binom{\alpha}{g}\binom{\beta}{h}$$

$$\times\{\cos(\alpha+\beta-2g-2h)x + j\sin(\alpha+\beta-2g-2h)x\} \tag{A.2c}$$

also

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \tag{A.2d}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[-\cos(\alpha+\beta)+\cos(\alpha-\beta)] \tag{A.2e}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{A.2f}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{A.2g}$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \tag{A.2h}$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \tag{A.2i}$$

And, some spherical trigonometry identities that we will use are

$$\cos a = \cos b\cos c + \sin b\sin c\cos A \tag{A.2j}$$

$$\cos b = \cos c\cos a + \sin c\sin a\cos B \tag{A.2k}$$

$$\cos c = \cos a\cos b + \sin a\sin b\cos C \tag{A.2l}$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{A.2m}$$

Now the geopotential is in a form compatible with the Inclination Function. The geopotential may be written as

$$V = \sum_{l=2}^{\infty}\sum_{m=0}^{l} V_{lm} \tag{A.3a}$$

where

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} P_l^m(\sin\theta)\left(C_{lm}\cos m\phi + S_{lm}\sin m\phi\right) \quad (A.3b)$$

Let

$$m\phi = m\left(\alpha - \theta_g\right) + m\Omega - m\Omega$$

where θ_g = prime meridian of Venus

α = right ascension

Ω = longitude of ascending node

or

$$m\theta = m(\alpha - \Omega) + m\left(\Omega - \theta_g\right) \quad (A.4)$$

Using eqn(A.2i) and eqn (A.2j), we have

$$\cos m\phi = \cos[m(\alpha - \Omega)]\cos\left[m\left(\Omega - \theta_g\right)\right]$$

$$-\sin[m(\alpha - \Omega)]\sin\left[m\left(\Omega - \theta_g\right)\right] \quad (A.5a)$$

$$\sin m\phi = \sin[m(\alpha - \Omega)]\cos\left[m\left(\Omega - \theta_g\right)\right]$$

$$+\cos[m(\alpha - \Omega)]\sin\left[m\left(\Omega - \theta_g\right)\right] \quad (A.5b)$$

Using eqn(A.2k) and eqn(A.2l), we get

$$\cos(\omega + f) = \cos(\alpha + \Omega)\cos\theta \quad (A.6a)$$

$$\cos\theta = \cos(\omega + f)\cos(\alpha - \Omega) + \sin(\omega + f)\sin(\alpha - \Omega)\cos i \quad (A.6b)$$

Eqn(A.2m) yields

$$\sin\theta = \sin(\omega + f)\sin i \tag{A.7}$$

Eliminating the terms containing *cos(ω+f)* from eqn(A.6a) and eqn(A.6b) gives

$$\cos\theta = \cos^2(\alpha - \Omega)\cos\theta + \sin(\omega + f)\cos i \sin(\alpha - \Omega)$$

$$\sin(\alpha - \Omega) = \sin(\omega + f)\frac{\cos i}{\cos\theta} \tag{A.8}$$

Using eqn(A.2a) and eqn(A.2b) for $(\alpha + \Omega)$ in eqn(A.5a) and eqn(A.5b) gives

$$\cos m\theta = Re\left\{\sum_{s=0}^{m}\binom{m}{s} j^s \cos^{m-s}(\alpha - \Omega)\sin^s(\alpha - \Omega)\right.$$

$$\left.\times\ \left[\cos\left(m\left(\Omega - \theta_g\right)\right) + j\sin\left(m\left(\Omega - \theta_g\right)\right)\right]\right\} \tag{A.9a}$$

$$\sin m\theta = Re\left\{\sum_{s=0}^{m}\binom{m}{s} j^s \cos^{m-s}(\alpha - \Omega)\sin^s(\alpha - \Omega)\right.$$

$$\left.\times\ \left[\sin\left(m\left(\Omega - \theta_g\right)\right) - j\cos\left(m\left(\Omega - \theta_g\right)\right)\right]\right\} \tag{A.9b}$$

Applying eqn(A.6a) and eqn(A.8) to the above equations, eqn.s(A.9), yields

$$\begin{bmatrix}\cos m\phi \\ \sin m\phi\end{bmatrix} = Re\left(\sum_{s=0}^{m}\binom{m}{s} j^s \frac{\cos^{m-s}(\omega + f)\sin^s(\omega + f)\cos^s i}{\cos^m\theta}\right.$$

$$\left.\times\begin{bmatrix}\cos\left[m\left(\Omega - \theta_g\right)\right] + j\sin\left[m\left(\Omega - \theta_g\right)\right] \\ \sin\left[m\left(\Omega - \theta_g\right)\right] - j\cos\left[m\left(\Omega - \theta_g\right)\right]\end{bmatrix}\right) \tag{A.10}$$

Now, substituting eqn(A.7) and eqn(A.10) into eqn(A.3), using eqn(A.12c), results in

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t}(i)$$

$$\times Re\Big(\left[C_{lm} - jS_{lm}\right]\cos\left[m\left(\Omega-\theta_g\right)\right] + \left[S_{lm} + jC_{lm}\right]\sin\left[m\left(\Omega-\theta_g\right)\right]$$

$$\times \sum_{s=0}^{m} \binom{m}{s} j^s \cos^{m-s}(\omega - f)\sin^{l-m-2t+s}(\omega+f)\cos^s(i)\Big) \qquad (A.11)$$

where $k =$ integer part of $(l-m)/2$

In eqn(A.2c), let $\alpha = l-m-2t+s$ and $\beta = m-s$, then define

$$A = \sin^{l-m-2t+s}(\omega+f)\cos^{m-s}(\omega+f)$$

$$= \frac{(-j)^{(l-m-2t+s)}}{2^{l-2t}} \sum_{g=0}^{(l-m-2t+s)} \sum_{h=0}^{(m-s)} (-1)^g \binom{l-m-2t+s}{g}\binom{m-s}{h}$$

$$\times \{\cos[(l-2t-2g-2h)(\omega+f)] + j\sin[(l-2t-2g-2h)(\omega+f)]\} \qquad (A.12)$$

Insert eqn(A.12) into eqn(A.11):

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t}(i) Re\left[\sum_{s=0}^{m}\binom{m}{s} j A_s \Delta \cos^s(i)\right] \qquad (A.13a)$$

where

$$\Delta = \left[C_{lm} - jS_{lm}\right]\cos\left[m\left(\Omega-\theta_g\right)\right] + \left[S_{lm} + jC_{lm}\right]\sin\left[m\left(\Omega-\theta_g\right)\right] \qquad (A.13b)$$

For notational convenience, let

$$\delta = m\left(\Omega - \theta_{g}\right) \tag{A.14a}$$

$$\lambda = (l - 2t - 2g - 2h)(\omega + f) \tag{A.14b}$$

Now, using eqn(A.2d-g), we have

$$\Delta\{\cos[(l-2t-2g-2h)(\omega+f)] + j\sin[(l-2t-2g-2h)(\omega+f)]\}$$

$$= \left[C_{lm} - jS_{lm}\right][\cos\delta\cos\lambda + j\cos\delta\sin\lambda]$$

$$+ \left[S_{lm} - jC_{lm}\right][\sin\delta\cos\lambda + j\sin\delta\sin\lambda]$$

$$+ \left[C_{lm} - jS_{lm}\right]\cos(\delta+\lambda) + \left[S_{lm} + jC_{lm}\right]\sin(\delta+\lambda) \tag{A.15}$$

Inserting eqn(A.15) into eqn(A.13b)

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} \sum_{t=0}^{k} T_{lmt} \sin^{l-m-2t}(i)$$

$$\times Re\left(\sum_{s=0}^{m} \binom{m}{s} \cos^s(i) \frac{j^s(-j)^{(l-m-2t+s)}}{2^{l-2t}} \sum_{g=0}^{(l-m-2t+s)} \sum_{h=0}^{(m-s)} \binom{l-m-2t+s}{g} \binom{m-s}{h} (-1)^g \right.$$

$$\left. \times \ \left[C_{lm} - jS_{lm}\right]\cos(\delta+\lambda) + \left[S_{lm} + jC_{lm}\right]\sin(\delta+\lambda) \right) \tag{A.16}$$

Now we will elliminate the complex numbers in the summations. Note

$$j^s(-j)^{l-m-2t+s} = j^s\left(\frac{1}{j}\right)^{l-m-2t+s} = j^{-l+m+2t} = j^{2\left(\frac{-l+m}{2}+t\right)} = (-1)^{-\left(\frac{l-m}{2}\right)+t} \tag{A.17a}$$

If $(l-m)$ is odd and k is the integer part of $(l-m)/2$, then

$$j^s(-j)^{l-m-2t+s} = (-1)^{t-k-1/2} = -j(-1)^{t-k} = -j(-1)^{t+k} \tag{A.17b}$$

However, if $(l-m)$ is even, then

$$j^{s}(-j)^{l-m-2t+s}=(-1)^{t-k}=(-1)^{t+k} \tag{A.17c}$$

Combining eqn($A.17b$) with eqn($A.17c$) yields

$$j^{s}(-j)^{l-m-2t+s}=\begin{cases}-j(-1)^{t+k}\ , & l-m \text{ even}\\(-1)^{t+k}\ , & l-m \text{ odd}\end{cases} \tag{A.17d}$$

Eqn($A.17d$) shows the real part of the geopotential (eqn($A.16$)) to depend on whether $(l-m)$ is even or odd. In other words:

$$V_{lm}=\frac{\mu R_e^l}{r^{l+1}}\sum_{t=0}^{k}T_{lmt}\sin^{l-m-2t}(i)$$

$$\times\sum_{s=0}^{m}\binom{m}{s}\frac{\cos^{s}(i)}{2^{l-2t}}\sum_{g=0}^{(l-m-2t+s)}\sum_{h=0}^{(m-s)}\binom{l-m-2t+s}{g}\binom{m-s}{h}(-1)^{g}$$

$$\times\begin{cases}C_{lm}\cos(\delta+\lambda)+S_{lm}\sin(\delta+\lambda)\ , & l-m \text{ even}\\-S_{lm}\cos(\delta+\lambda)+C_{lm}\sin(\delta+\lambda)\ , & l-m \text{ odd}\end{cases} \tag{A.18a}$$

where $\delta+\lambda=(l-2t-2g-2h)(\omega+f)+m(\Omega-\theta_g)$

Let $p=t+g+h$ and define

$$H=\delta+\lambda=(l-2p)(\omega+f)+m(\Omega-\theta_g) \tag{A.18b}$$

Using eqn($A.18b$)

$$V_{lm}=\frac{\mu R_e^l}{r^{l+1}}\sum_{t=0}^{k}T_{lmt}(-1)^{k+1}\sin^{l-m-2t}(i)$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \frac{\cos^s(i)}{2^{l-2t}} \sum_{g=0}^{(l-m-2t+s)} \sum_{h=0}^{(m-s)} \binom{l-m-2t+s}{g} \binom{m-s}{h} (-1)^t$$

$$\times \begin{cases} C_{lm}\cos H + S_{lm}\sin H \;, & l-m \text{ even} \\ -S_{lm}\cos H + C_{lm}\sin H \;, & l-m \text{ odd} \end{cases} \qquad (A.19)$$

The summations of the geopotential (eqn(A.19)) must now be interchanged. To do this we must note some relationships. If $t<0$ then $T_{lmt}=0$ (since $|(-t)|\rightarrow\infty$). Also, if $t>k$ then $T_{lmt}=0$ (since $|(l-m-2t)|\rightarrow\infty$). Hence, the sum on t can be $-\infty<t<\infty$, since T_{lmt} will be zero for all values of t except $0\leq t\leq k$. Also, the summation on p can be $-\infty<p<\infty$, since the binomial coefficient

$$\binom{m-s}{p-t-g}$$

will be zero if $p<t+g$ or $p>t+g+m-s$. Therefore, the limits on the summations are independent of each other and may be interchanged. So, substituting for T_{lmt}, eqn(A.12c), the geopotential (eqn(A.19)) may be written as

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} \sum_{p=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \frac{(2l-2t)!\sin^{(l-m-2t)}(i)}{t!(l-t)!(l-m-2t)!2^{2l-2t}}$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^s(i) \sum_{g=0}^{(l-m-2t+s)} \binom{l-m-2t+s}{g} \binom{m-s}{p-t-g} (-1)^{g-k}$$

$$\times \begin{cases} C_{lm}\cos H + S_{lm}\sin H \;, & l-m \text{ even} \\ -S_{lm}\cos H + C_{lm}\sin H \;, & l-m \text{ odd} \end{cases} \qquad (A.20)$$

Before the Inclination Function can be introduced into the geopotential (eqn(A.20)) the limits of the summations over p, t, and g must be evaluated. The minimum of $p=t+g+h$ will be zero since the minimum values of t, g, and h are zero. The maximum value of p will be

$$(p)_{max} = (t+g+h)_{max} = (t+l-m-2t+s+m-s)_{max}$$

$$(p)_{max} = (-t+l)_{max} = l \qquad (A.21a)$$

since $t \geq 0$. Also,

$$(t)_{max} = (p-g-h)_{max} = p$$

or, if $p>k$ then $(t)_{max}=k$, as can be seen from eqn(2.2.12c) for T_{lmt}. Therefore

$$(t)_{max} = \begin{cases} p & p \leq k \\ k & p > k \end{cases} \qquad (A.21b)$$

Now to look at the index g in eqn(A.20). The range of g will be limited by the binomial coefficients involving g. So, look at

$$\binom{l-m-2t+s}{g}\binom{m-s}{p-t-g}$$

The above binomial coefficients have the following four conditions:

$$\begin{cases} g \geq 0 \\ m-s \geq p-t-g \\ l-m-2t+s \geq g \\ p-t \geq g \end{cases}$$

The first two conditions give the lower limit for g and the last two give the upper limit of g. These limits are

$$g_L = \max[0, p-t-m+s] \tag{A.21c}$$

$$g_U = \min[l-m-2t+s, p-t] \tag{A.21d}$$

The Inclination Function, $F_{lmp}(i)$(7:110), is defined as

$$F_{lmp}(i) = \sum_{t=0}^{(t)_{max}} \frac{(2l-2t)!}{t!(l-t)!(l-m-2t)!2^{2l-2t}} \sin^{(l-m-2t)}(i)$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^{s}(i) \sum_{g=g_L}^{g_U} \binom{l-m-2t+s}{g} \binom{m-s}{p-t-g} (-1)^{g-k} \tag{A.22}$$

Inserting the $F_{lmp}(i)$ (eqn($A.22$)) into the geopotential (eqn($A.20$)) yeilds

$$V_{lm} = \frac{\mu R_e^l}{r^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \begin{cases} C_{lm}\cos H + S_{lm}\sin H \ , & l-m \text{ even} \\ -S_{lm}\cos H + C_{lm}\sin H \ , & l-m \text{ odd} \end{cases} \tag{A.23}$$

The geopotential (eqn($A.23$), is now ready to be modified to allow the inclusion of the Eccentricity Function, $G_{lpq}(e)$. The portion of the geopotential to be transformed is

$$\frac{1}{r^{l+1}} \begin{cases} C_{lm}\cos H + S_{lm}\sin H \ , & l-m \text{ even} \\ -S_{lm}\cos H + C_{lm}\sin H \ , & l-m \text{ odd} \end{cases}$$

To do this, examine

$$\frac{1}{r^{l+1}}\begin{cases}\cos H\\ \sin H\end{cases} \tag{A.24}$$

Only long period terms (those which do not contain M) are of interest. So, the geopotential (eqn(A.23)) may be averaged with respect to M. To do this, integrate with respect to M from 0 to 2π and then divide by 2π. This is the Method of Averaging (17:11). The following identities will be used

$$dM = \frac{r^2 df}{a^2\sqrt{1-e^2}} \tag{A.25a}$$

$$r = \frac{a\left(1-e^2\right)}{1+e\cos f} \tag{A.25b}$$

$$\frac{1}{r} = \frac{1}{a\left(1-e^2\right)} + \frac{e}{a\left(1-e^2\right)}\cos f \tag{A.25c}$$

From eqn(A.25b) and eqn(A.25c) deduce

$$\frac{r^2}{r^{l+1}} = \left(\frac{1+e\cos f}{a\left(1-e^2\right)}\right)^{l-1} \tag{A.25d}$$

The binomial expansion of $(1+e\cos f)^{l-1}$ is

$$\left(1+e\cos f\right)^{l-1} = \sum_{b=0}^{l-1}\binom{l-1}{b}e^{l-1}\cos^{l-1}f \tag{A.25f}$$

Integrating eqn(A.24):

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r^{l+1}}\begin{cases}\cos H\\ \sin H\end{cases} dM =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{r^2}{r^{l+1}a^2(1-e^2)^{1/2}}\begin{cases}\cos H\\ \sin H\end{cases} df$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1+e\cos f)^{l-1}}{\left(a^2\sqrt{1-e^2}\right)\left[a(1-e^2)\right]^{l-1}}\begin{cases}\cos H\\ \sin H\end{cases} df$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(1+e\cos f)^{l-1}}{a^{l+1}(1-e^2)^{l-1/2}}\begin{cases}\cos H\\ \sin H\end{cases} df$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{a^{l+1}(1-e^2)^{l-1/2}}\sum_{b=0}^{l-1}\left[\binom{l-1}{b}e^b\cos^b f\begin{cases}\cos H\\ \sin H\end{cases}\right]df \qquad (A.26)$$

Here the long period terms are of interest. This is when (see Section 2.5)

$$(l-2p)\pm(b-2d)=0$$

Recall H is given by eqn($A.18b$). Expand the sin H and cos H using the identities of eqn.s($A.2h$) and ($A.2i$) and let

$$B=(l-2p)\omega+m(\Omega-\theta)$$

so

$$\sin[(l-2p)f+B]=\sin(l-2p)f\cos B+\cos(l-2p)f\sin B$$

$$\cos[(l-2p)f+B]=\cos(l-2p)f\cos B-\sin(l-2p)f\sin B$$

The sine function is odd, and, therefore (over the limits of integration) may be ignored. So

$$\sin[(l-2p)f+B]=\cos(l-2p)f\sin B$$

$$\cos[(l-2p)f+B]=\cos(l-2p)f\cos B$$

Inserting these identities into eqn(A.26), then pulling out variables <u>not</u> dependent on f, and interchanging the integral and summation results in

$$\frac{1}{a^{l-1}}\frac{1}{(1-e^2)^{l-1/2}}\sum_{b=0}^{l-1}\binom{l-1}{b}e^b\times$$

$$\times\frac{1}{2\pi}\int_0^{2\pi}\cos^b f\cos(l-2p)f\,df\begin{cases}\sin(l-2p)\omega+m(\Omega-\theta)\\ \cos(l-2p)\omega+m(\Omega-\theta)\end{cases}\qquad (A.27)$$

In order to integrate eqn(A.27) Hansen's Coefficient will be introduced. Hansen's coefficient may be defined as (2)

$$X_0^{n,m}=(1-e^2)^{n+3/2}\sum_{b=0}^{\infty}(-1)^b\binom{n+b+1}{b}e^b\frac{1}{2\pi}\int_0^{2\pi}\cos^b f\cos mf\,df\qquad (A.28)$$

let

$$n=-(l+1)$$

$$m=(l-2p)$$

Then

$$X_0^{-(l+1),(l-2p)}=\frac{1}{(1-e^2)^{l-1/2}}\sum_{b=0}^{\infty}(-1)^b\binom{-l+b}{b}e^b\frac{1}{2\pi}\int_0^{2\pi}\cos^b f\cos(l-2p)f\,df$$

but

$$\binom{-l+b}{b} = (-1)^b \binom{l-1}{b}$$

using(1:11)

$$\binom{-r}{b} = (-1)^b \binom{b+r-1}{b}$$

So

$$X_0^{-(l+1),(l-2p)} = \frac{1}{(1-e^2)^{l-1/2}} \sum_{b=0}^{\infty} \binom{l-1}{b} e^b \frac{1}{\pi} \int_0^{2\pi} \cos^b f \cos(l-2p) f \, df$$

Insert this Hansen's Coefficient into eqn(A.27) and the integration becomes

$$\frac{1}{2\pi} \int_0^{2\pi} \begin{Bmatrix} \sin H \\ \cos H \end{Bmatrix} dM =$$

$$= \frac{1}{a^{l+1}} X_0^{-(l+1),(l-2p)} \begin{Bmatrix} \sin[(l-2p)f + m(\Omega-\theta)] \\ \cos[(l-2p)f + m(\Omega-\theta)] \end{Bmatrix} \qquad (A.28a)$$

or

$$= \frac{1}{a^{l+1}} G_{lp(2p-l)} \begin{Bmatrix} \sin[(l-2p)f + m(\Omega-\theta)] \\ \cos[(l-2p)f + m(\Omega-\theta)] \end{Bmatrix} \qquad (A.28b)$$

Where

$$G_{lp(2p-l)} = X_0^{-(l+1),(l-2p)}$$

to follow the notation of Kaula (12:110).

Comparing eqn(A.28) and eqn(A.28a) shows n=-(l-1). Recall from eqn(A.1) that l ≥ 0; therefore, $n \leq 2$ and Hansen's Coefficient for $n \leq 2$ may be used which is defined as (2)

$$X_0^{n,m} = \left(\frac{e}{2}\right)^{|m|} \sum_{d=0}^{-\left[\frac{n+|m|+2}{2}\right]} \binom{-n-2}{|m|+2d} \binom{|m|+2d}{d} \left(\frac{e}{2}\right)^{2d} \qquad (A.29)$$

let

$$n = -(l+1)$$

$$m = l - 2p \Rightarrow |m| = l - 2p' \begin{cases} p' = p \ , & p \leq l/2 \\ p' = l - p \ , & p > l/2 \end{cases}$$

So, Hansen's Coefficient becomes

$$X_0^{-(l+1),(l-2p)} = \frac{1}{(1-e^2)^{l-1/2}} \sum_{d=0}^{p'-1} \binom{l-1}{2d+l-2p'} \binom{2d+l-p'}{d} \left(\frac{e}{2}\right)^{2d+l-2p'}$$

and the Eccentricity Function, $G_{lp(2p-l)}(e)$, is

$$G_{lp(2p-l)}(e) = \frac{1}{(1-e^2)^{l-1/2}} \sum_{d=0}^{p'-1} \binom{l-1}{2d+l-2p'} \binom{2d+l-p'}{d} \left(\frac{e}{2}\right)^{2d+l-2p'} \qquad (A.30)$$

So that the integration of eqn(A.26) is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^{l+1}} \begin{Bmatrix} \sin H \\ \cos H \end{Bmatrix} dM = \frac{1}{a^{l+1}(1-e^2)^{l-1/2}}$$

$$\times\sum_{d=0}^{p'-1}\binom{l-1}{2d+l-2p'}\binom{2d+1-2p'}{d}\left(\frac{e}{2}\right)^{2d+l-2p'}\begin{cases}\cos H\\ \sin H\end{cases} \qquad (A.31)$$

Note that the factor of 1/2 in the summation over d has been dropped. This is because the two terms satisfying the long period variation, $(l-2p)*(b-2d)=0$, are symmetric in the binomial expansion and combine to make the $(2d+l-2p)$ substitution for b.

Now the disturbing function for a nonspherical planet is given by

$$R=\sum_{l=2}^{\infty}\sum_{m=0}^{l}V_{lm} \qquad (A.32a)$$

$$\text{where } V_{lm}=\frac{\mu R_e^l}{a^{l+1}}\sum_{p=0}^{l}F_{lmp}(i)\sum_{q=-\infty}^{\infty}G_{lpq}(e)S_{lmpq}(\omega,M,\Omega,\theta) \qquad (A.32b)$$

$$\text{and } S_{lmpq}=\begin{cases}C_{lm}\cos\phi+S_{lm}\sin\phi\ , & l-m \text{ even}\\ -S_{lm}\cos\phi+C_{lm}\sin\phi\ , & l-m \text{ odd}\end{cases} \qquad (A.32c)$$

$$\text{where } \phi=(l-2p)\omega+(l-2p+q)M+m(\Omega-\theta) \qquad (A.32d)$$

Appendix B

Conversion of the Geopotential into the Modified Orbital Elements

In Appendix A the geopotential was found to be

$$R = \sum_{l=2}^{\infty} \sum_{m=0}^{l} V_{lm} \tag{A.32a}$$

$$\text{where } V_{lm} = \frac{\mu R_e^l}{a^{l+1}} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S_{lmpq}(\omega, M, \Omega, \theta) \tag{A.32b}$$

$$\text{and } S_{lmpq} = \begin{cases} C_{lm}\cos\phi + S_{lm}\sin\phi \ , \ l-m \text{ even} \\ -S_{lm}\cos\phi + C_{lm}\sin\phi \ , \ l-m \text{ odd} \end{cases} \tag{A.32c}$$

$$\text{where } \phi = (l-2p)\omega + (l-2p+q)M + m(\Omega-\theta) \tag{A.32d}$$

Define

$$J_{lm} = \begin{cases} \left(C_{lm}^2 + S_{lm}^2\right)^{1/2}, m \neq 0 \\ C_{l0}, m = 0 \end{cases}$$

$$\lambda_{lm} = \begin{cases} \dfrac{1}{m}\tan^{-1}\left(\dfrac{S_{lm}}{C_{lm}}\right), m \neq 0 \\ 0, m = 0 \end{cases}$$

then V_{lm} (eqn($A.32b$)) may be written as

$$V_{lm} = \frac{\mu R_e^l}{a^{l+1}} J_{lm} \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) S_{lmpq}^{*} \tag{B.1a}$$

where

$$S'_{lmpq}(\omega, M, \Omega, \theta, \lambda_N) = \begin{cases} \cos\phi', & l-m \text{ even} \\ \sin\phi', & l-m \text{ odd} \end{cases} \tag{B.1b}$$

$$\phi' = (l-2p)\omega + (l-2p+q)M + m(\Omega-\theta) - m\lambda_{lm} \tag{B.1c}$$

For Venus only the long term effects are of interest so

$$q = 2p - l$$

as stated in Section 2.6. Replace S''_{lmpq} with

$$S'_{lmq} = \left.\begin{cases} \cos(\xi - q\omega), & l-m \text{ even} \\ \sin(\xi - q\omega), & l-m \text{ odd} \end{cases}\right] \tag{B.2a}$$

where

$$\xi = m(\lambda_N - \lambda_{nm}) \tag{B.2b}$$

$$q = 2p - l \tag{B.2c}$$

$$m = 0, 1, 2, \ldots, l \tag{B.2d}$$

Multiply and divide eqn(B.2) by

$$e^{|q|}$$

and use the trigonometric identities of eqn.s(A.2h) and (A.2i), to yeild

$$S'_{lmq} = \frac{1}{e^{|q|}}\begin{cases} \cos\xi\, e^{|q|}\cos q\omega + \sin\xi\, e^{|q|}\sin q\omega, & l-m \text{ even} \\ \sin\xi\, e^{|q|}\cos q\omega - \cos\xi\, e^{|q|}\sin q\omega, & l-m \text{ odd} \end{cases} \tag{B.3}$$

and define

$$G'_{lpq} = G_{lpq} / e^{|q|} \tag{B.4}$$

Note ξ (eqn($B.2b$)) is a function of λ_N. Also, note that $e^{|q|}\sin q\omega$ and $e^{|q|}\cos q\omega$ are related to h and k. From eqn($A.2b$)

$$e^{|q|}\sin q\omega = e^{|q|} Re\left[\sum_{i=1}^{q}\binom{q}{i}(j)^{i-1}\cos^{q-i}\omega\sin^{i}\omega\right]$$

But $e^{|q|}$ can be written as

$$e^{|q|} = e^{|q|-i+i} = e^{|q|-i}e^{i}$$

so

$$e^{|q|}\sin q\omega = Re\left[\sum_{i=1}^{q}\binom{q}{i}(j)^{i-1}e^{|q|-i}\cos^{q-i}\omega\, e^{i}\sin^{i}\omega\right]$$

Recall

$$h = e\sin\omega \tag{2.1.2a}$$

$$k = e\cos\omega \tag{2.1.2b}$$

so

$$e^{|q|}\sin q\omega = Re\left[\sum_{i=0}^{|q|}\binom{|q|}{i}(j)^{i-1}k^{|q|-i}h^{i}\right]$$

To extract the real part of the above eqn, let $i = 2j-3$ so

$$e^{|q|}\sin q\omega = \frac{q}{|q|}\sum_{j=0}^{n_1}(-1)^j\binom{|q|}{2j-3}k^{|q|-2j+3}h^{2j-3} \tag{B.5a}$$

where $n_1 = |q|+3)/2$

Similarly, $e^{|q|}\cos q\omega$ may be expressed as

$$e^{|q|}\cos q\omega = \sum_{j=0}^{n_2}(-1)^j\binom{|q|}{2j}k^{|q|-2j}h^{2j} \tag{B.5b}$$

where $n_2 = |q|/2$. So that the geopotential becomes

$$V_{lm} = \frac{\mu R_e^l}{a^{l+1}}J_{lm}\sum_{p=0}^{l}F_{lmp}(i)\sum_{q=-\infty}^{\infty}G'_{lpq}(e)S_{lmq}(h,k,\lambda_N) \tag{B.6a}$$

where

$$S_{lmq} = e^{|q|}S'_{lmq} \tag{B.6b}$$

Appendix C

Useful Derivatives for the

Geopotential Calculations

When calculating Lagrange's Equations for the Geopotential in Section II the following derivatives are useful(11a).

$$\frac{\partial S_{lmq}}{\partial h} =$$

$$\begin{cases} \dfrac{\partial S_1}{\partial h}\cos\xi + \dfrac{q}{|q|}\dfrac{\partial S_2}{\partial h}\sin\xi, & l-m \text{ even} \\ \dfrac{\partial S_1}{\partial h}\sin\xi + \dfrac{q}{|q|}\dfrac{\partial S_2}{\partial h}\cos\xi, & l-m \text{ odd...} \end{cases} \qquad (C.1a)$$

$$\frac{\partial S_{lmq}}{\partial k} =$$

$$\begin{cases} \dfrac{\partial S_2}{\partial h}\cos\xi + \dfrac{q}{|q|}\dfrac{\partial S_1}{\partial h}\sin\xi, & l-m \text{ even} \\ \dfrac{\partial S_2}{\partial h}\sin\xi + \dfrac{q}{|q|}\dfrac{\partial S_1}{\partial h}\cos\xi, & l-m \text{ odd} \end{cases} \qquad (C.1b)$$

$$\frac{\partial S_{lmq}}{\partial \Omega} = m \begin{cases} -S_1\sin\xi + \dfrac{q}{|q|}S_2\cos\xi \;, & l-m \text{ odd} \\ S_1\cos\xi + \dfrac{q}{|q|}S_2\sin\xi \;, & l-m \text{ even} \end{cases} \qquad (C.1c)$$

$$\frac{\partial S_{lmq}}{\partial \lambda_N} = \frac{\partial S_{lmq}}{\partial \Omega} \qquad (C.1d)$$

where

$$S_1 = \sum_{j=0}^{n_1} (-1)^j \binom{|q|}{2j} k^{|q|-2j} h^{2j} \tag{C.2a}$$

$$S_2 = \sum_{j=0}^{n_2} (-1)^j \binom{|q|}{2j-3} k^{|q|-2j+3} h^{2j-3} \tag{C.2b}$$

$$\frac{1}{e}\frac{\partial G'}{\partial e} = \frac{2(l-1/2)}{1-e^2} G' + \frac{1}{2^{l-2p'+1}!\left(1-e^2\right)^{l-1/2}}$$

$$\times \sum_{k=1}^{p'-1} \frac{k(l-1)!}{(2p'-2k-1)!(k+l-2p')!k!} \left(\frac{e}{2}\right)^{2k-2} \tag{C.3}$$

$$\frac{\partial F_{lmp}}{\partial i} = \cos i \sum_{t=0}^{T} \frac{(2l-2t)!(l-m-2t)}{t!(l-t)!(l-m-2t)!2^{2l-2t}} \sin^{(l-m-2t-1)} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^s i \sum_{c} \binom{l-m-2t+s}{c} \binom{m-s}{p-t-c} n(-1)^{c-k}$$

$$-\sin i \sum_{t=0}^{T} \frac{(2l-2t)!}{t!(l-t)!(l-m-2t)!2^{2l-2t}} \sin^{(l-m-2t)} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} s \cos^{s-1} i \sum_{c} \binom{l-m-2t+s}{c} \binom{m-s}{p-t-c} n(-1)^{c-k} \tag{C.4}$$

Appendix D

Supplemental Tables for Section III and IV

On the following pages are tabulations of the data for inclinations from 2.5 degrees to 15.5 degrees. The data ranges are:

$2.5° \leq i \leq 15.5°$

$0.02 \leq e \leq 0.10$

$0° \leq \Omega \leq 330°$

$\omega = 53°$

$M = 0$

TABLE D.1: INSERTION WINDOW HYPERSURFACE (i=2.5°, 3.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
2.50	0.02	0	238.54	3.50	0.02	0	241.48
2.50	0.02	15	226.78	3.50	0.02	15	229.82
2.50	0.02	60	208.55	3.50	0.02	60	209.24
2.50	0.02	105	231.39	3.50	0.02	105	228.35
2.50	0.02	150	269.02	3.50	0.02	150	265.59
2.50	0.02	195	297.34	3.50	0.02	195	295.09
2.50	0.02	240	306.26	3.50	0.02	240	305.57
2.50	0.02	285	293.42	3.50	0.02	285	294.30
2.50	0.02	330	263.24	3.50	0.02	330	265.49
2.50	0.04	0	194.76	3.50	0.04	0	195.14
2.50	0.04	15	182.09	3.50	0.04	15	182.38
2.50	0.04	60	166.15	3.50	0.04	60	165.19
2.50	0.04	105	193.90	3.50	0.04	105	190.92
2.50	0.04	150	239.40	3.50	0.04	150	236.14
2.50	0.04	195	273.19	3.50	0.04	195	270.89
2.50	0.04	240	281.74	3.50	0.04	240	280.58
2.50	0.04	285	262.73	3.50	0.04	285	262.34
2.50	0.04	330	223.85	3.50	0.04	330	223.94
2.50	0.06	0	174.06	3.50	0.06	0	173.68
2.50	0.06	15	162.21	3.50	0.06	15	161.74
2.50	0.06	60	148.21	3.50	0.06	60	146.98
2.50	0.06	105	175.00	3.50	0.06	105	172.46
2.50	0.06	150	220.96	3.50	0.06	150	217.95
2.50	0.06	195	255.65	3.50	0.06	195	253.49
2.50	0.06	240	263.54	3.50	0.06	240	262.23
2.50	0.06	285	242.77	3.50	0.06	285	241.92
2.50	0.06	330	202.63	3.50	0.06	330	202.07
2.50	0.08	0	161.21	3.50	0.08	0	160.56
2.50	0.08	15	149.98	3.50	0.08	15	149.43
2.50	0.08	60	137.20	3.50	0.08	60	136.09
2.50	0.08	105	162.59	3.50	0.08	105	160.38
2.50	0.08	150	207.21	3.50	0.08	150	204.54
2.50	0.08	195	241.52	3.50	0.08	195	239.41
2.50	0.08	240	248.70	3.50	0.08	240	247.41
2.50	0.08	285	227.54	3.50	0.08	285	226.53
2.50	0.08	330	188.26	3.50	0.08	330	187.43
2.50	0.10	0	151.89	3.50	0.10	0	151.17
2.50	0.10	15	141.45	3.50	0.10	15	140.82
2.50	0.10	60	129.57	3.50	0.10	60	128.58
2.50	0.10	105	153.42	3.50	0.10	105	151.53
2.50	0.10	150	196.26	3.50	0.10	150	193.92
2.50	0.10	195	229.38	3.50	0.10	195	227.49
2.50	0.10	240	235.95	3.50	0.10	240	234.69
2.50	0.10	285	214.98	3.50	0.10	285	213.99
2.50	0.10	330	177.45	3.50	0.10	330	176.55

TABLE D.2: INSERTION WINDOW HYPERSURFACE (i = 4.5° , 5.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
4.50	0.02	0	245.21	5.50	0.02	0	249.71
4.50	0.02	15	233.64	5.50	0.02	15	238.35
4.50	0.02	60	211.10	5.50	0.02	60	214.14
4.50	0.02	105	226.59	5.50	0.02	105	225.80
4.50	0.02	150	263.04	5.50	0.02	150	261.38
4.50	0.02	195	293.62	5.50	0.02	195	292.83
4.50	0.02	240	305.57	5.50	0.02	240	306.16
4.50	0.02	285	295.77	5.50	0.02	285	297.73
4.50	0.02	330	268.43	5.50	0.02	330	272.06
4.50	0.04	0	196.20	5.50	0.04	0	197.83
4.50	0.04	15	183.53	5.50	0.04	15	185.26
4.50	0.04	60	165.38	5.50	0.04	60	166.44
4.50	0.04	105	189.19	5.50	0.04	105	188.33
4.50	0.04	150	233.83	5.50	0.04	150	232.30
4.50	0.04	195	269.35	5.50	0.04	195	268.49
4.50	0.04	240	280.01	5.50	0.04	240	280.01
4.50	0.04	285	262.54	5.50	0.04	285	263.11
4.50	0.04	330	244.71	5.50	0.04	330	225.96
4.50	0.06	0	173.96	5.50	0.06	0	174.81
4.50	0.06	15	162.12	5.50	0.06	15	163.06
4.50	0.06	60	146.89	5.50	0.06	60	147.55
4.50	0.06	105	170.95	5.50	0.06	105	170.39
4.50	0.06	150	215.89	5.50	0.06	150	214.57
4.50	0.06	195	251.98	5.50	0.06	195	251.14
4.50	0.06	240	261.57	5.50	0.06	240	261.29
4.50	0.06	285	241.64	5.50	0.06	285	241.74
4.50	0.06	330	202.07	5.50	0.06	330	202.54
4.50	0.08	0	160.47	5.50	0.08	0	161.02
4.50	0.08	15	149.52	5.50	0.08	15	150.17
4.50	0.08	60	136.00	5.50	0.08	60	136.55
4.50	0.08	105	159.18	5.50	0.08	105	158.72
4.50	0.08	150	202.79	5.50	0.08	150	201.69
4.50	0.08	195	238.12	5.50	0.08	195	237.29
4.50	0.08	240	246.68	5.50	0.08	240	246.40
4.50	0.08	285	226.07	5.50	0.08	285	225.98
4.50	0.08	330	187.15	5.50	0.08	330	187.34
4.50	0.10	0	151.08	5.50	0.10	0	151.44
4.50	0.10	15	140.82	5.50	0.10	15	141.36
4.50	0.10	60	128.49	5.50	0.10	60	129.12
4.50	0.10	105	150.54	5.50	0.10	105	150.27
4.50	0.10	150	192.30	5.50	0.10	150	191.40
4.50	0.10	195	226.32	5.50	0.10	195	225.60
4.50	0.10	240	234.06	5.50	0.10	240	233.79
4.50	0.10	285	213.54	5.50	0.10	285	213.36
4.50	0.10	330	176.19	5.50	0.10	330	176.28

TABLE D.3: INSERTION WINDOW HYPERSURFACE (i=6.5° , 7.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
6.50	0.02	0	255.01	7.50	0.02	0	260.98
6.50	0.02	15	243.93	7.50	0.02	15	250.40
6.50	0.02	60	218.26	7.50	0.02	60	223.55
6.50	0.02	105	226.00	7.50	0.02	105	227.17
6.50	0.02	150	260.40	7.50	0.02	150	260.10
6.50	0.02	195	292.64	7.50	0.02	195	292.93
6.50	0.02	240	307.14	7.50	0.02	240	308.61
6.50	0.02	285	300.28	7.50	0.02	285	303.22
6.50	0.02	330	276.27	7.50	0.02	330	281.17
6.50	0.04	0	200.14	7.50	0.04	0	202.73
6.50	0.04	15	187.66	7.50	0.04	15	190.54
6.50	0.04	60	168.26	7.50	0.04	60	170.76
6.50	0.04	105	188.33	7.50	0.04	105	189.00
6.50	0.04	150	231.43	7.50	0.04	150	231.14
6.50	0.04	195	268.01	7.50	0.04	195	268.01
6.50	0.04	240	280.30	7.50	0.04	240	280.97
6.50	0.04	285	264.07	7.50	0.04	285	265.42
6.50	0.04	330	227.59	7.50	0.04	330	229.70
6.50	0.06	0	176.03	7.50	0.06	0	177.72
6.50	0.06	15	164.47	7.50	0.06	15	166.44
6.50	0.06	60	148.96	7.50	0.06	60	150.93
6.50	0.06	105	170.48	7.50	0.06	105	171.24
6.50	0.06	150	213.91	7.50	0.06	150	213.72
6.50	0.06	195	250.67	7.50	0.06	195	250.67
6.50	0.06	240	261.48	7.50	0.06	240	261.85
6.50	0.06	285	242.21	7.50	0.06	285	242.86
6.50	0.06	330	203.29	7.50	0.06	330	204.51
6.50	0.08	0	161.94	7.50	0.08	0	163.23
6.50	0.08	15	151.27	7.50	0.08	15	152.84
6.50	0.08	60	137.84	7.50	0.08	60	139.59
6.50	0.08	105	159.00	7.50	0.08	105	159.92
6.50	0.08	150	201.14	7.50	0.08	150	201.14
6.50	0.08	195	236.92	7.50	0.08	195	236.92
6.50	0.08	240	246.49	7.50	0.08	240	246.77
6.50	0.08	285	226.25	7.50	0.08	285	226.71
6.50	0.08	330	187.89	7.50	0.08	330	188.72
6.50	0.10	0	152.25	7.50	0.10	0	153.42
6.50	0.10	15	142.35	7.50	0.10	15	143.79
6.50	0.10	60	130.29	7.50	0.10	60	132.00
6.50	0.10	105	150.72	7.50	0.10	105	151.62
6.50	0.10	150	191.04	7.50	0.10	150	191.04
6.50	0.10	195	225.33	7.50	0.10	195	225.33
6.50	0.10	240	233.88	7.50	0.10	240	234.15
6.50	0.10	285	213.54	7.50	0.10	285	213.99
6.50	0.10	330	176.64	7.50	0.10	330	177.36

TABLE D.4: INSERTION WINDOW HYPERSURFACE (i = 8.5°, 9.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
8.50	0.02	0	267.75	9.50	0.02	0	275.29
8.50	0.02	15	257.55	9.50	0.02	15	265.49
8.50	0.02	60	229.82	9.50	0.02	60	237.27
8.50	0.02	105	229.33	9.50	0.02	105	232.56
8.50	0.02	150	260.49	9.50	0.02	150	261.47
8.50	0.02	195	293.62	9.50	0.02	195	294.70
8.50	0.02	240	310.47	9.50	0.02	240	312.63
8.50	0.02	285	306.65	9.50	0.02	285	310.67
8.50	0.02	330	286.66	9.50	0.02	330	292.83
8.50	0.04	0	205.90	9.50	0.04	0	209.64
8.50	0.04	15	193.99	9.50	0.04	15	198.02
8.50	0.04	60	173.93	9.50	0.04	60	177.77
8.50	0.04	105	190.44	9.50	0.04	105	192.46
8.50	0.04	150	231.34	9.50	0.04	150	232.01
8.50	0.04	195	268.30	9.50	0.04	195	268.97
8.50	0.04	240	281.93	9.50	0.04	240	283.08
8.50	0.04	285	266.95	9.50	0.04	285	268.87
8.50	0.04	330	232.20	9.50	0.04	330	235.08
8.50	0.06	0	179.79	9.50	0.06	0	182.33
8.50	0.06	15	168.79	9.50	0.06	15	171.61
8.50	0.06	60	153.38	9.50	0.06	60	156.38
8.50	0.06	105	172.55	9.50	0.06	105	174.43
8.50	0.06	150	214.01	9.50	0.06	150	214.66
8.50	0.06	195	250.95	9.50	0.06	195	251.51
8.50	0.06	240	262.51	9.50	0.06	240	263.36
8.50	0.06	285	243.80	9.50	0.06	285	245.03
8.50	0.06	330	206.02	9.50	0.06	330	207.80
8.50	0.08	0	164.89	9.50	0.08	0	166.91
8.50	0.08	15	154.77	9.50	0.08	15	157.07
8.50	0.08	60	141.89	9.50	0.08	60	144.56
8.50	0.08	105	161.21	9.50	0.08	105	163.05
8.50	0.08	150	201.50	9.50	0.08	150	202.24
8.50	0.08	195	237.20	9.50	0.08	195	237.75
8.50	0.08	240	247.32	9.50	0.08	240	248.15
8.50	0.08	285	227.45	9.50	0.08	285	228.37
8.50	0.08	330	189.82	9.50	0.08	330	191.20
8.50	0.10	0	154.95	9.50	0.10	0	156.75
8.50	0.10	15	145.59	9.50	0.10	15	147.75
8.50	0.10	60	134.16	9.50	0.10	60	136.77
8.50	0.10	105	153.06	9.50	0.10	105	154.86
8.50	0.10	150	191.58	9.50	0.10	150	192.39
8.50	0.10	195	225.69	9.50	0.10	195	226.32
8.50	0.10	240	234.78	9.50	0.10	240	235.50
8.50	0.10	285	214.62	9.50	0.10	285	215.52
8.50	0.10	330	178.35	9.50	0.10	330	179.61

TABLE D.5: INSERTION WINDOW HYPERSURFACE (i = 10.5° . 11.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
10.50	0.02	0	283.62	11.50	0.02	0	292.74
10.50	0.02	15	274.31	11.50	0.02	15	283.92
10.50	0.02	60	245.60	11.50	0.02	60	255.10
10.50	0.02	105	236.78	11.50	0.02	105	242.27
10.50	0.02	150	263.04	11.50	0.02	150	265.30
10.50	0.02	195	296.26	11.50	0.02	195	298.32
10.50	0.02	240	315.28	11.50	0.02	240	318.41
10.50	0.02	285	315.18	11.50	0.02	285	320.37
10.50	0.02	330	299.79	11.50	0.02	330	307.44
10.50	0.04	0	213.86	11.50	0.04	0	218.76
10.50	0.04	15	202.63	11.50	0.04	15	207.91
10.50	0.04	60	182.38	11.50	0.04	60	187.56
10.50	0.04	105	195.05	11.50	0.04	105	198.31
10.50	0.04	150	233.06	11.50	0.04	150	234.60
10.50	0.04	195	269.93	11.50	0.04	195	271.27
10.50	0.04	240	284.52	11.50	0.04	240	286.34
10.50	0.04	285	271.08	11.50	0.04	285	273.67
10.50	0.04	330	238.44	11.50	0.04	330	242.28
10.50	0.06	0	185.15	11.50	0.06	0	188.44
10.50	0.06	15	174.81	11.50	0.06	15	178.47
10.50	0.06	60	159.96	11.50	0.06	60	164.09
10.50	0.06	105	176.78	11.50	0.06	105	179.70
10.50	0.06	150	215.79	11.50	0.06	150	217.20
10.50	0.06	195	252.26	11.50	0.06	195	253.39
10.50	0.06	240	264.48	11.50	0.06	240	265.80
10.50	0.06	285	246.53	11.50	0.06	285	248.22
10.50	0.06	330	209.96	11.50	0.06	330	212.41
10.50	0.08	0	169.30	11.50	0.08	0	172.06
10.50	0.08	15	159.83	11.50	0.08	15	162.96
10.50	0.08	60	147.78	11.50	0.08	60	151.46
10.50	0.08	105	165.35	11.50	0.08	105	168.11
10.50	0.08	150	203.25	11.50	0.08	150	204.72
10.50	0.08	195	238.58	11.50	0.08	195	239.59
10.50	0.08	240	249.16	11.50	0.08	240	250.26
10.50	0.08	285	229.56	11.50	0.08	285	230.94
10.50	0.08	330	192.86	11.50	0.08	330	194.88
10.50	0.10	0	158.91	11.50	0.10	0	161.43
10.50	0.10	15	150.18	11.50	0.10	15	153.06
10.50	0.10	60	139.83	11.50	0.10	60	143.25
10.50	0.10	105	157.11	11.50	0.10	105	159.81
10.50	0.10	150	193.47	11.50	0.10	150	194.91
10.50	0.10	195	227.13	11.50	0.10	195	228.21
10.50	0.10	240	236.49	11.50	0.10	240	237.66
10.50	0.10	285	216.51	11.50	0.10	285	217.77
10.50	0.10	330	181.05	11.50	0.10	330	182.85

TABLE D.6: INSERTION WINDOW HYPERSURFACE (i = 12.5° , 13.5°)							
i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
12.50	0.02	0	302.73	13.50	0.02	0	313.61
12.50	0.02	15	294.40	13.50	0.02	15	305.67
12.50	0.02	60	265.49	13.50	0.02	60	276.96
12.50	0.02	105	248.93	13.50	0.02	105	256.97
12.50	0.02	150	268.33	13.50	0.02	150	272.16
12.50	0.02	195	300.87	13.50	0.02	195	303.91
12.50	0.02	240	322.04	13.50	0.02	240	326.25
12.50	0.02	285	326.15	13.50	0.02	285	332.82
12.50	0.02	330	315.96	13.50	0.02	330	325.27
12.50	0.04	0	224.14	13.50	0.04	0	230.28
12.50	0.04	15	213.77	13.50	0.04	15	220.39
12.50	0.04	60	193.61	13.50	0.04	60	200.42
12.50	0.04	105	202.25	13.50	0.04	105	206.95
12.50	0.04	150	236.62	13.50	0.04	150	239.11
12.50	0.04	195	272.81	13.50	0.04	195	274.73
12.50	0.04	240	288.36	13.50	0.04	240	290.76
12.50	0.04	285	276.65	13.50	0.04	285	280.01
12.50	0.04	330	246.60	13.50	0.04	330	251.59
12.50	0.06	0	192.20	13.50	0.06	0	196.43
12.50	0.06	15	182.70	13.50	0.06	15	187.31
12.50	0.06	60	168.70	13.50	0.06	60	173.87
12.50	0.06	105	183.08	13.50	0.06	105	186.93
12.50	0.06	150	218.99	13.50	0.06	150	221.15
12.50	0.06	195	254.71	13.50	0.06	195	256.31
12.50	0.06	240	267.40	13.50	0.06	240	269.18
12.50	0.06	285	250.20	13.50	0.06	285	252.45
12.50	0.06	330	215.32	13.50	0.06	330	218.52
12.50	0.08	0	175.10	13.50	0.08	0	178.60
12.50	0.08	15	166.45	13.50	0.08	15	170.41
12.50	0.08	60	155.50	13.50	0.08	60	160.10
12.50	0.08	105	171.24	13.50	0.08	105	174.82
12.50	0.08	150	206.47	13.50	0.08	150	208.59
12.50	0.08	195	240.88	13.50	0.08	195	242.44
12.50	0.08	240	251.74	13.50	0.08	240	253.30
12.50	0.08	285	232.51	13.50	0.08	285	234.44
12.50	0.08	330	197.18	13.50	0.08	330	199.76
12.50	0.10	0	164.22	13.50	0.10	0	167.37
12.50	0.10	15	156.21	13.50	0.10	15	159.81
12.50	0.10	60	147.12	13.50	0.10	60	151.35
12.50	0.10	105	162.87	13.50	0.10	105	166.38
12.50	0.10	150	196.71	13.50	0.10	150	198.87
12.50	0.10	195	229.56	13.50	0.10	195	231.09
12.50	0.10	240	239.01	13.50	0.10	240	240.54
12.50	0.10	285	219.30	13.50	0.10	285	221.01
12.50	0.10	330	184.92	13.50	0.10	330	187.26

TABLE D.7: INSERTION WINDOW HYPERSURFACE (i = 14.5° . 15.5°)

i(°)	e	Ω(°)	Hp (Km)	i(°)	e	Ω(°)	Hp (Km)
14.50	0.02	0	325.17	15.50	0.02	0	337.72
14.50	0.02	15	317.82	15.50	0.02	15	330.76
14.50	0.02	60	289.40	15.50	0.02	60	302.73
14.50	0.02	105	266.37	15.50	0.02	105	277.35
14.50	0.02	150	276.96	15.50	0.02	150	282.94
14.50	0.02	195	307.63	15.50	0.02	195	312.04
14.50	0.02	240	331.15	15.50	0.02	240	336.93
14.50	0.02	285	340.36	15.50	0.02	285	348.79
14.50	0.02	330	335.56	15.50	0.02	330	346.73
14.50	0.04	0	237.10	15.50	0.04	0	244.78
14.50	0.04	15	227.78	15.50	0.04	15	235.94
14.50	0.04	60	208.01	15.50	0.04	60	216.55
14.50	0.04	105	213.42	15.50	0.04	105	218.76
14.50	0.04	150	242.18	15.50	0.04	150	245.74
14.50	0.04	195	277.03	15.50	0.04	195	279.82
14.50	0.04	240	293.54	15.50	0.04	240	296.71
14.50	0.04	285	283.85	15.50	0.04	285	288.36
14.50	0.04	330	257.16	15.50	0.04	330	263.40
14.50	0.06	0	201.22	15.50	0.06	0	206.49
14.50	0.06	15	192.57	15.50	0.06	15	198.40
14.50	0.06	60	179.70	15.50	0.06	60	186.18
14.50	0.06	105	191.45	15.50	0.06	105	196.52
14.50	0.06	150	223.78	15.50	0.06	150	226.79
14.50	0.06	195	258.28	15.50	0.06	195	260.54
14.50	0.06	240	271.25	15.50	0.06	240	273.70
14.50	0.06	285	255.08	15.50	0.06	285	258.09
14.50	0.06	330	222.18	15.50	0.06	330	226.41
14.50	0.08	0	182.55	15.50	0.08	0	186.88
14.50	0.08	15	174.82	15.50	0.08	15	179.70
14.50	0.08	60	165.26	15.50	0.08	60	170.87
14.50	0.08	105	178.96	15.50	0.08	105	183.56
14.50	0.08	150	211.07	15.50	0.08	150	213.92
14.50	0.08	195	244.28	15.50	0.08	195	246.40
14.50	0.08	240	255.23	15.50	0.08	240	257.35
14.50	0.08	285	236.56	15.50	0.08	285	239.04
14.50	0.08	330	202.70	15.50	0.08	330	206.10
14.50	0.10	0	170.97	15.50	0.10	0	174.93
14.50	0.10	15	163.86	15.50	0.10	15	168.27
14.50	0.10	60	156.12	15.50	0.10	60	161.34
14.50	0.10	105	170.34	15.50	0.10	105	174.66
14.50	0.10	150	201.30	15.50	0.10	150	204.09
14.50	0.10	195	232.89	15.50	0.10	195	235.05
14.50	0.10	240	242.43	15.50	0.10	240	244.50
14.50	0.10	285	222.99	15.50	0.10	285	225.24
14.50	0.10	330	189.87	15.50	0.10	330	192.93

Appendix E

Supplemental Figures for Section III and IV

The following pages contain 3D graphs and contour mappings of the tabulated data from Appendix D.

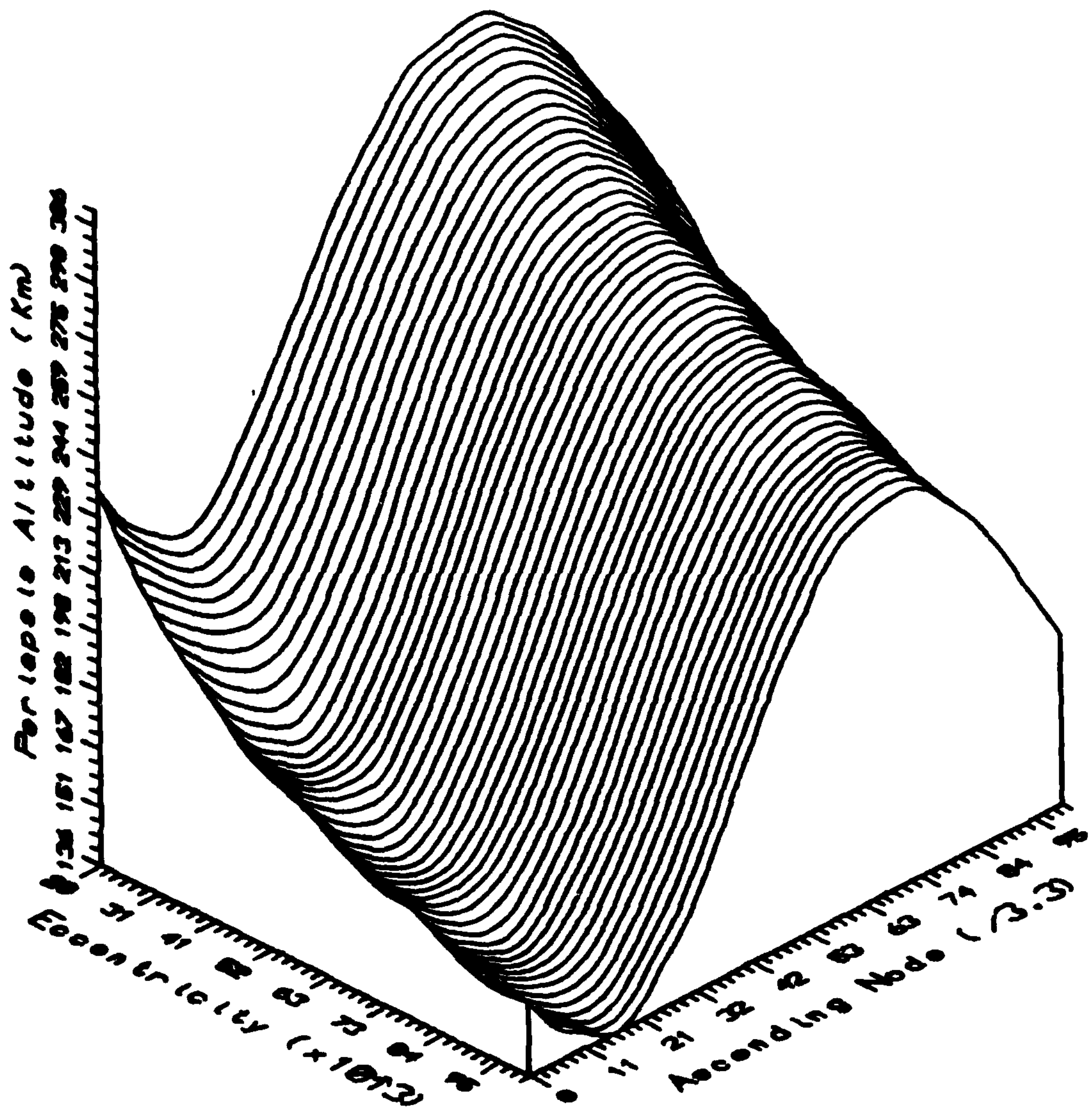

Fig E.1: Periapsis Altitude (i = 0.5 deg)

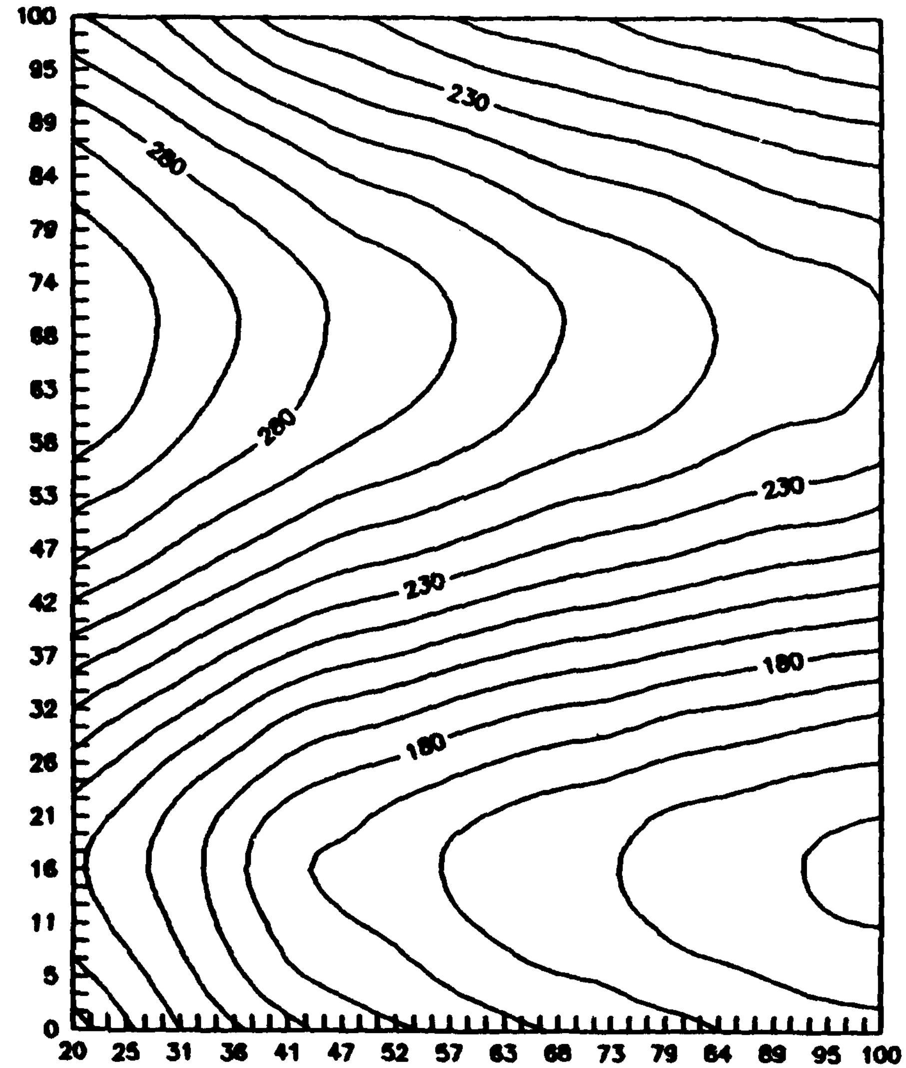

Fig E.2: Periapsis Contour (i = 0.5 deg)

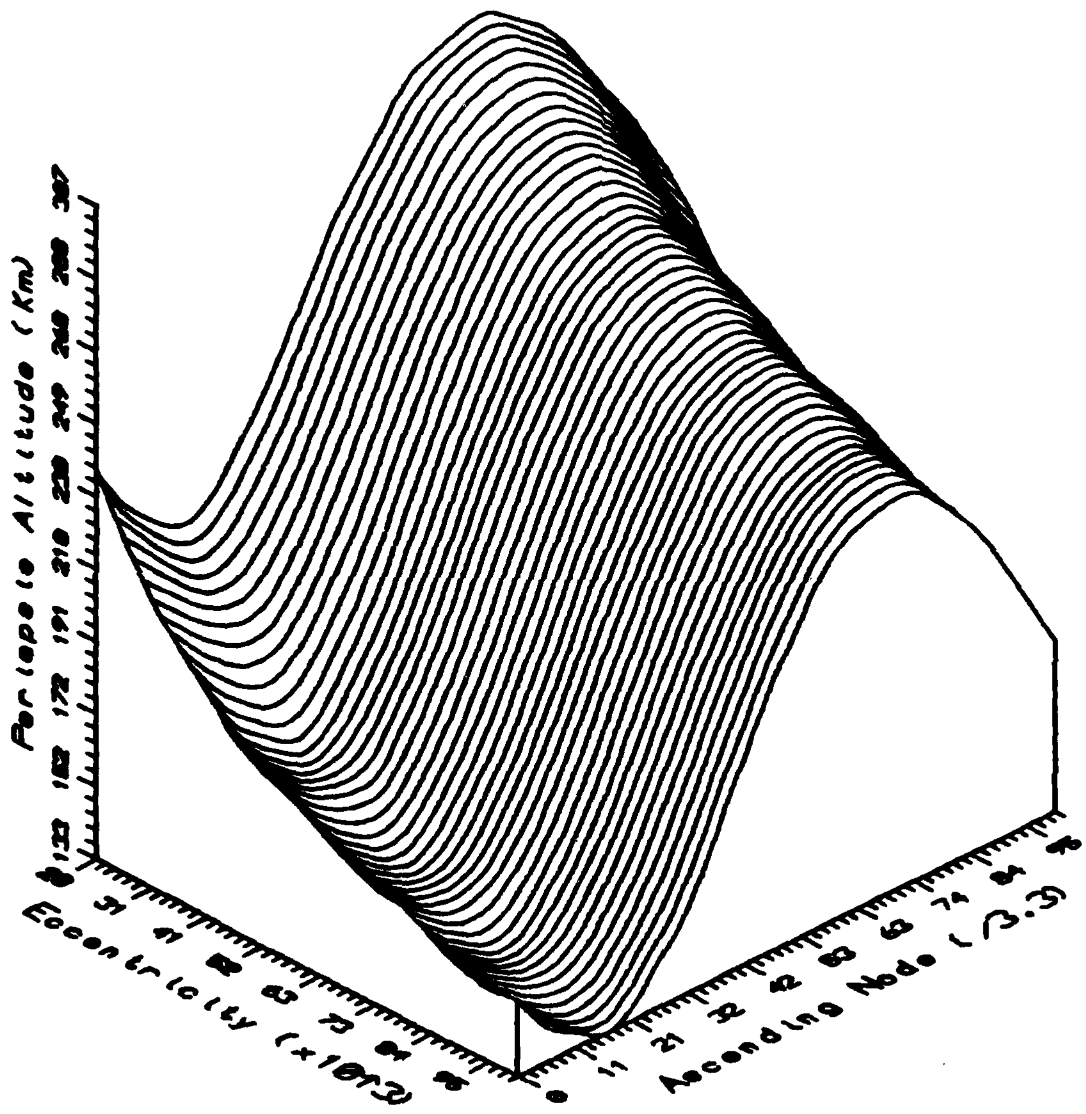

Fig E.3: Periapsis Altitude (i = 1.5 deg)

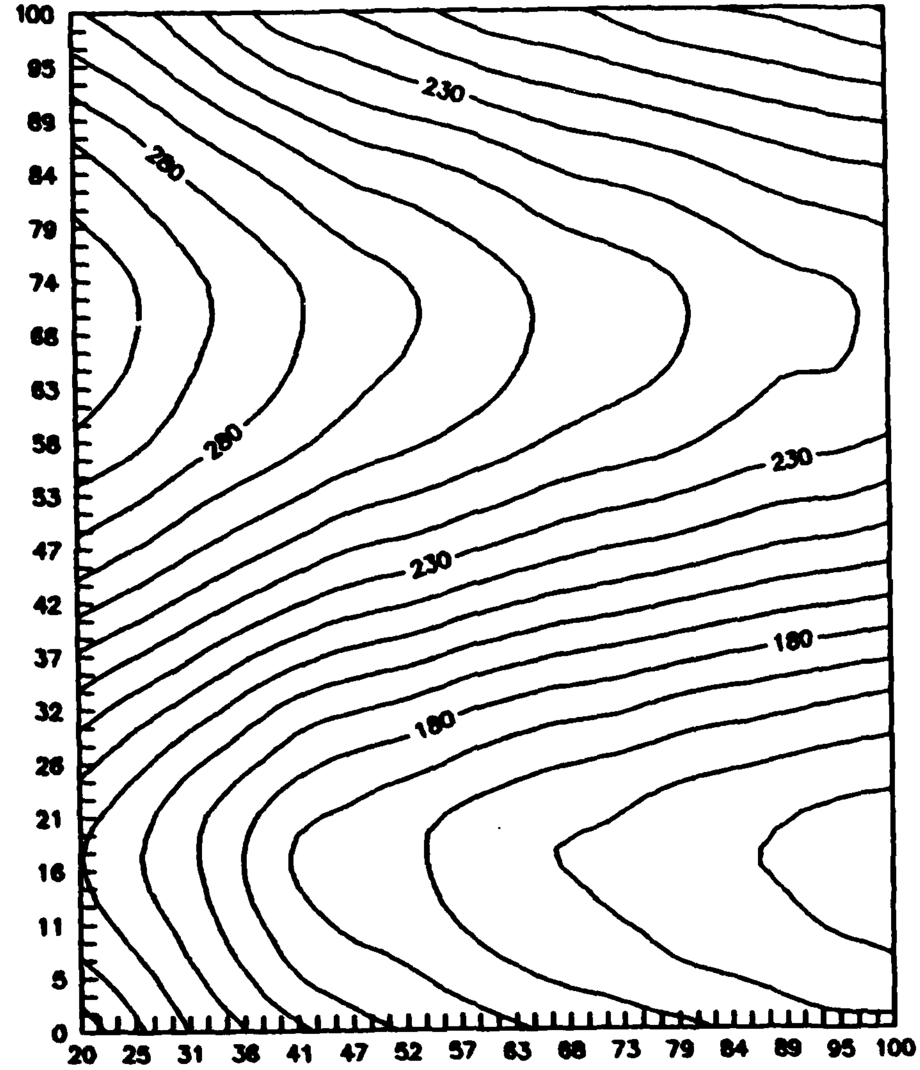

Fig E.4: Periapsis Contour (i = 1.5 deg)

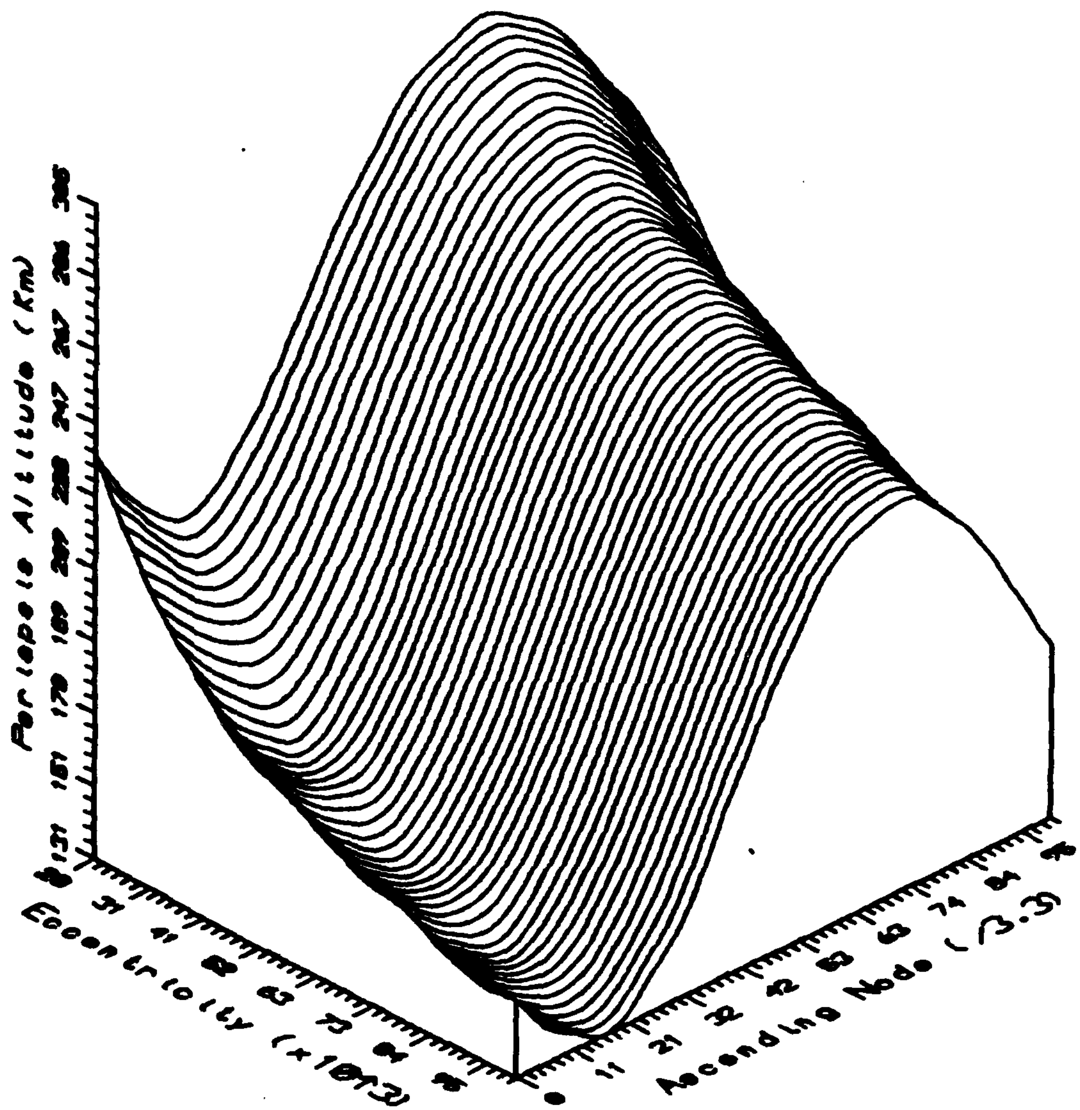

Fig E.5: Periapsis Altitude (i = 2.5 deg)

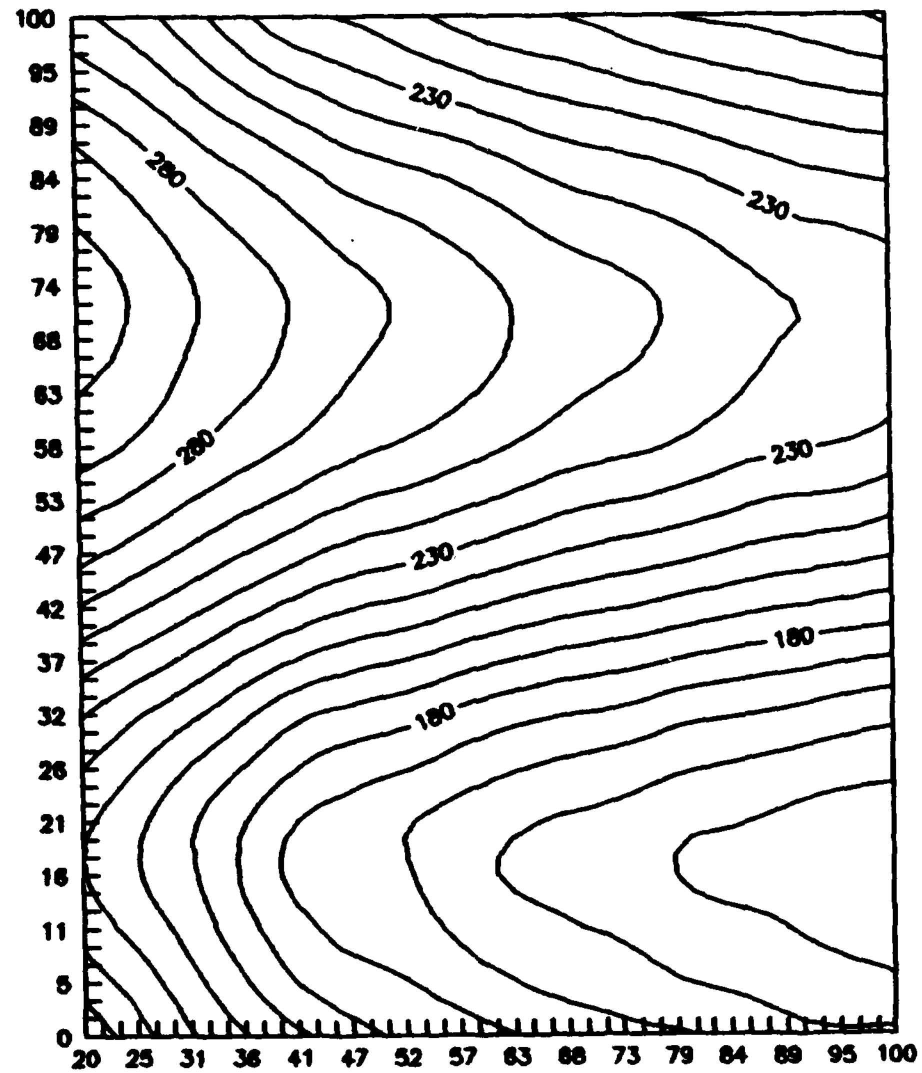

Fig E.6: Periapsis Contour (i = 2.5 deg)

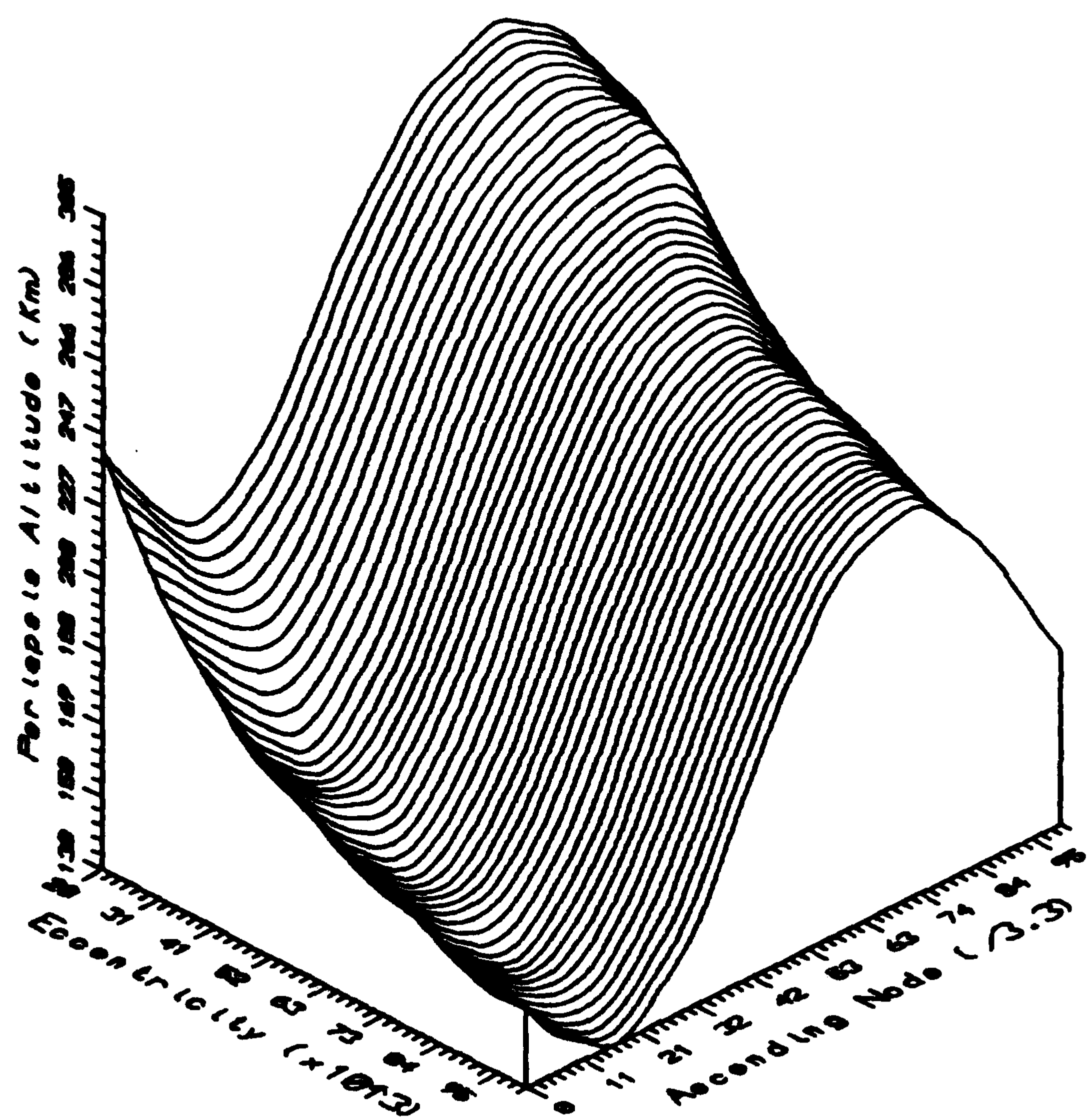

Fig E.7: Periapsis Altitude (i = 3.5 deg)

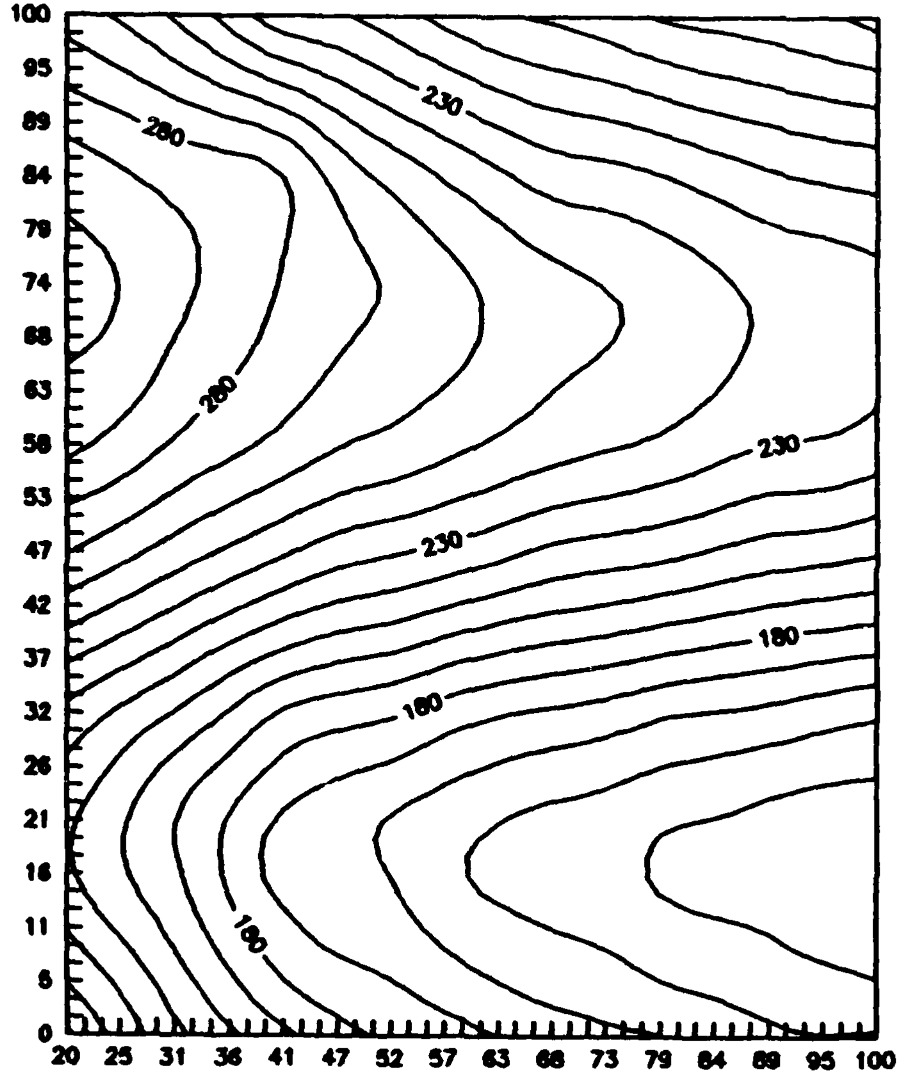

Fig E.8: Periapsis Contour (i = 3.5 deg)

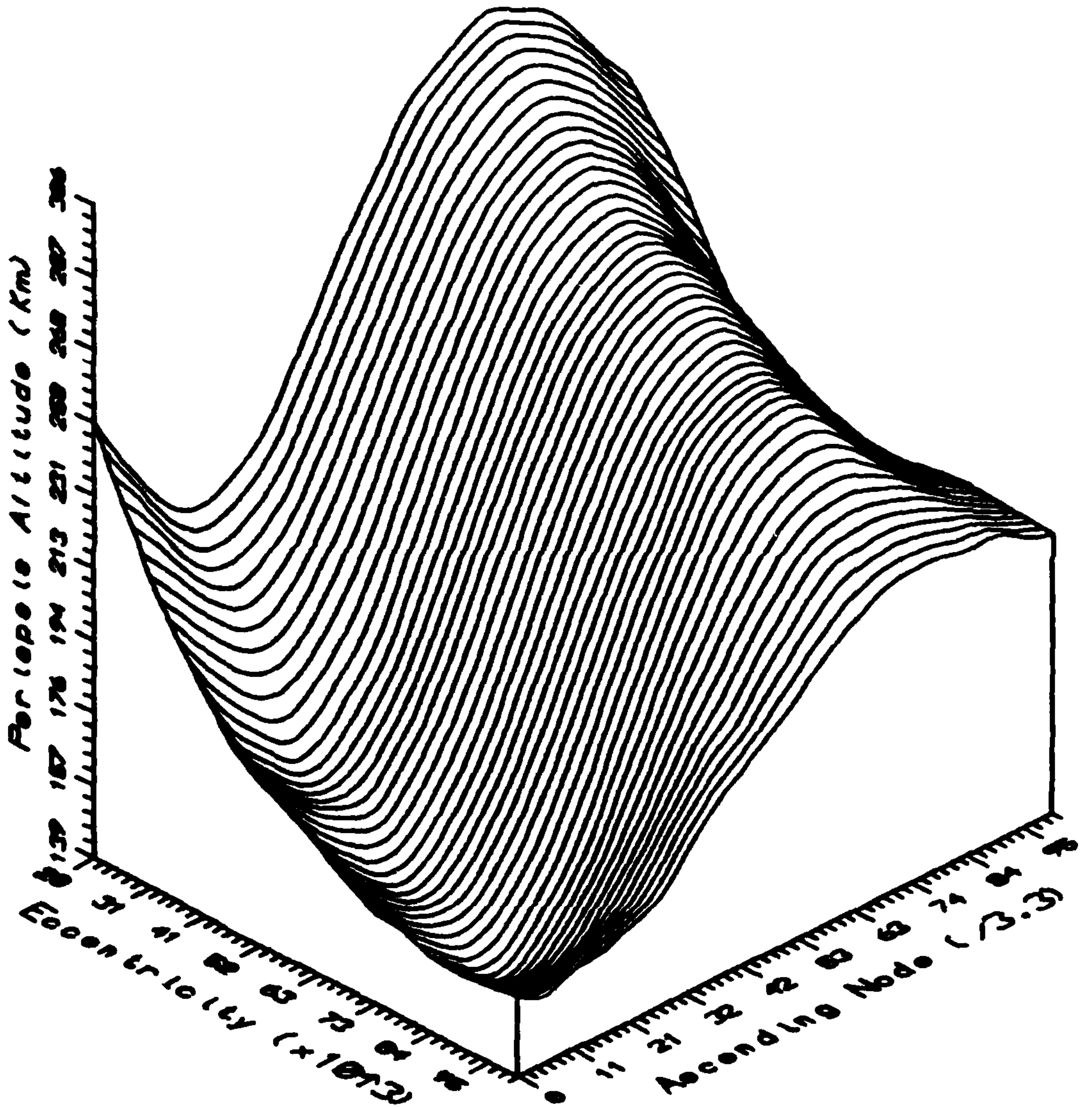

Fig E.9: Periapsis Altitude (i = 4.5 deg)

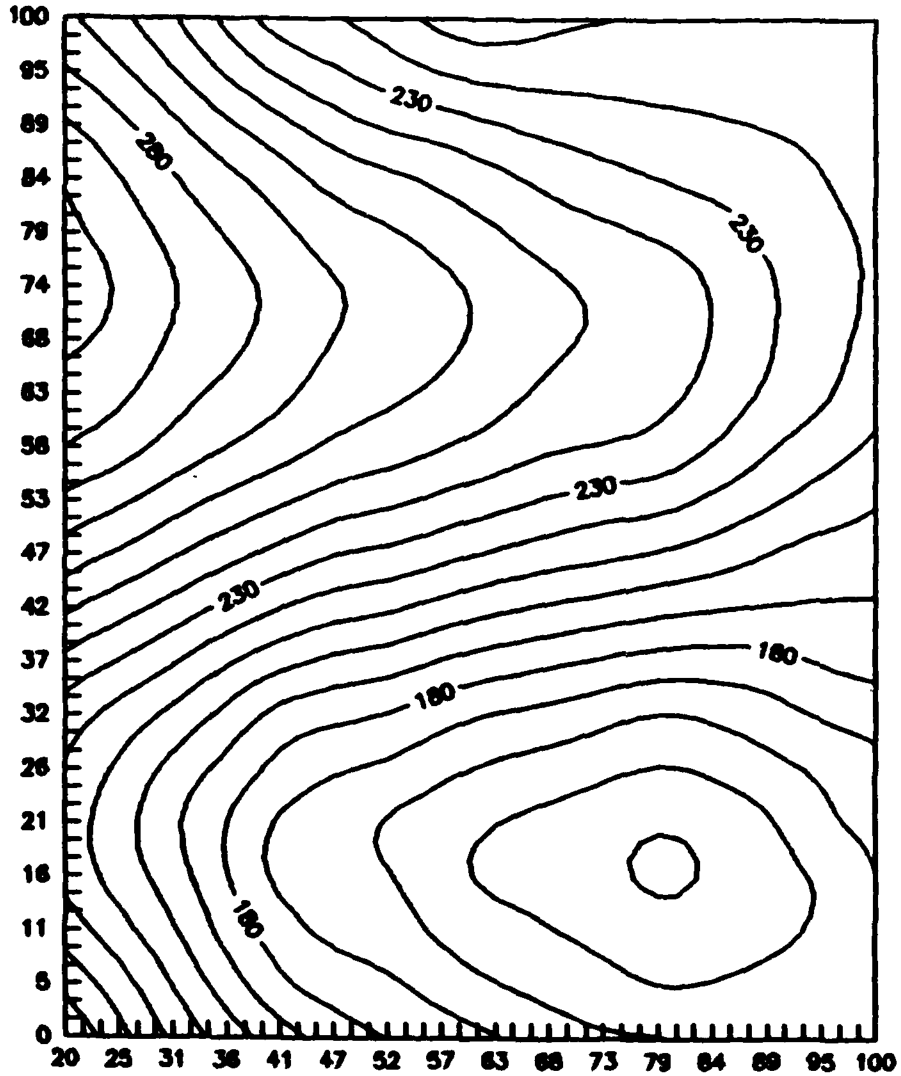

Fig E.10: Periapsis Altitude (i = 4.5 deg)

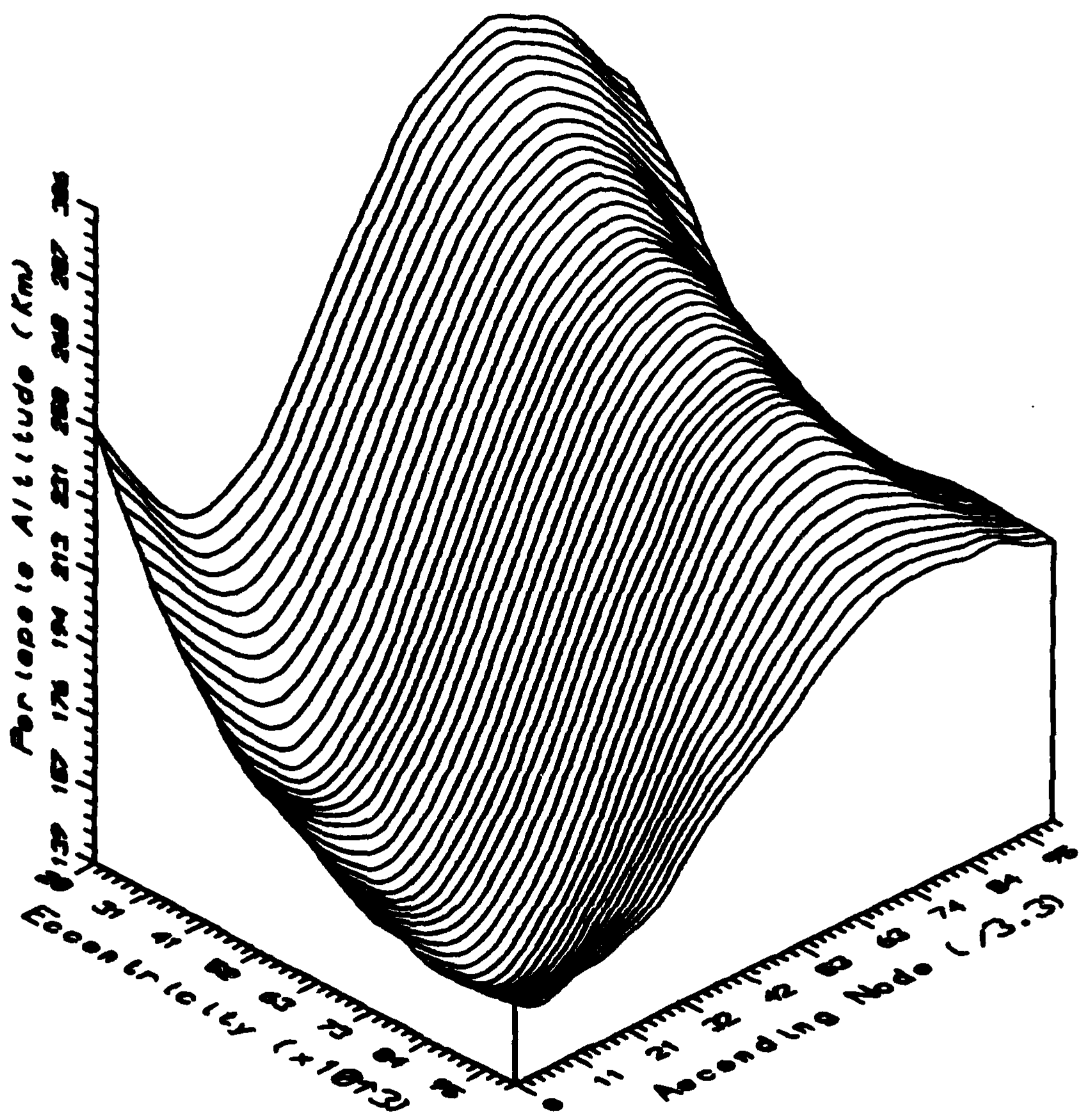

Fig E.11: Perlapsis Altitude (i = 5.5 deg)

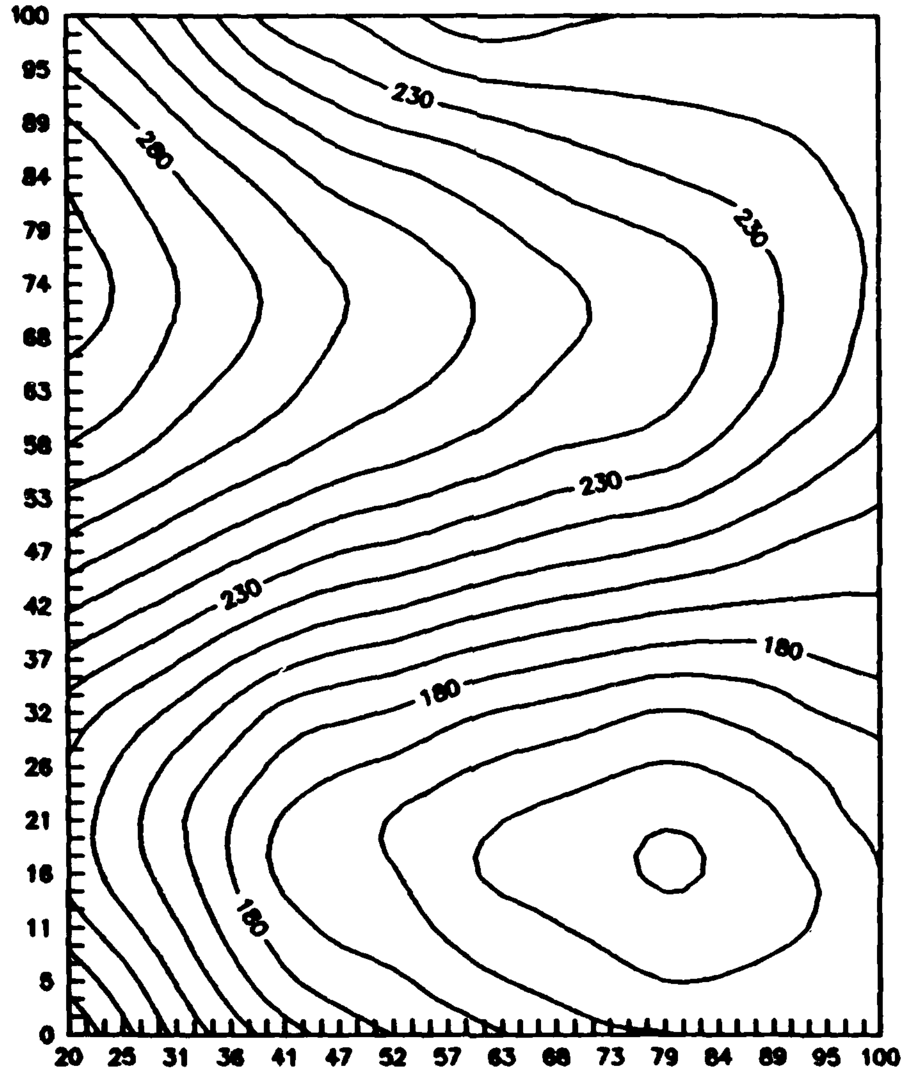

Fig E.12: Periapsis Altitude (i = 5.5 deg)

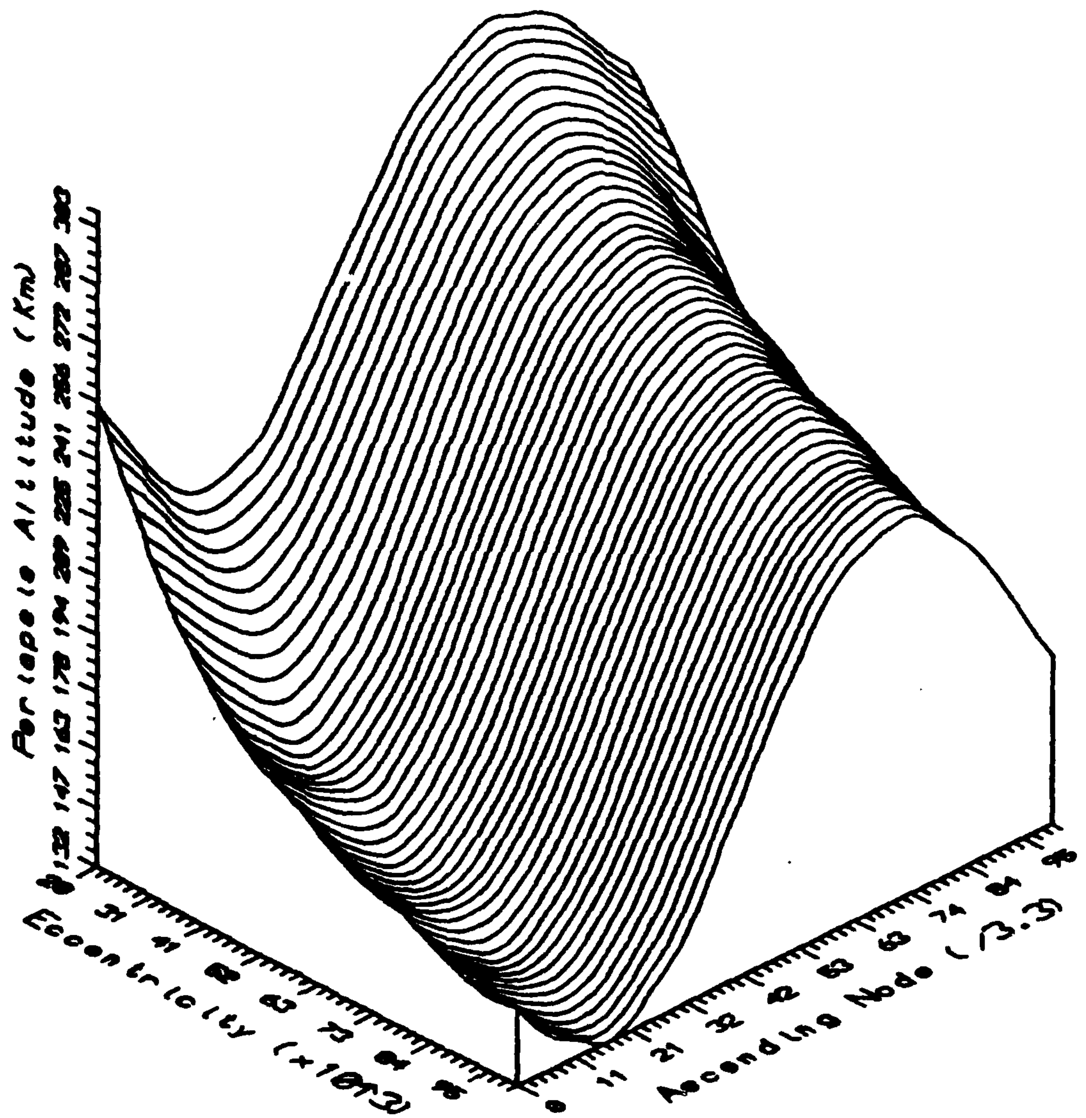

Fig E.13: Periapsis Altitude (i = 6.5 deg)

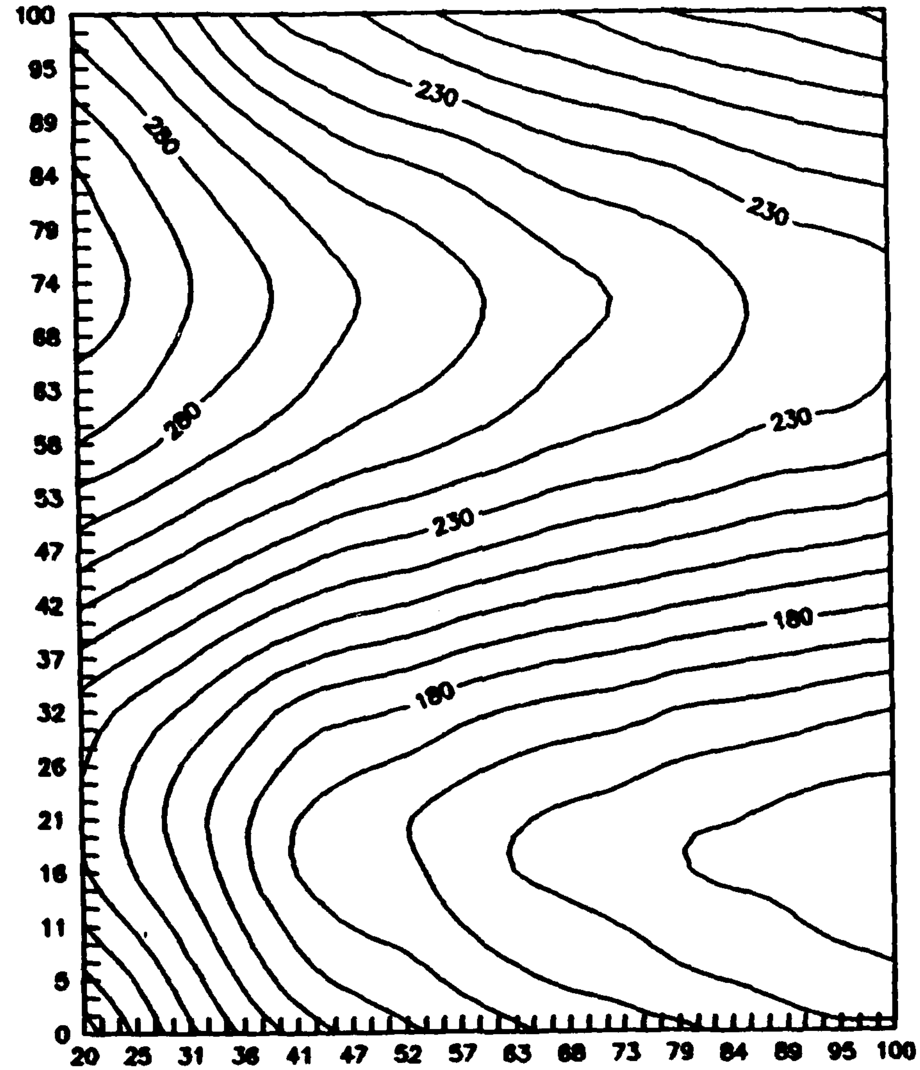

Fig E.14: Periapsis Altitude (i = 6.5 deg)

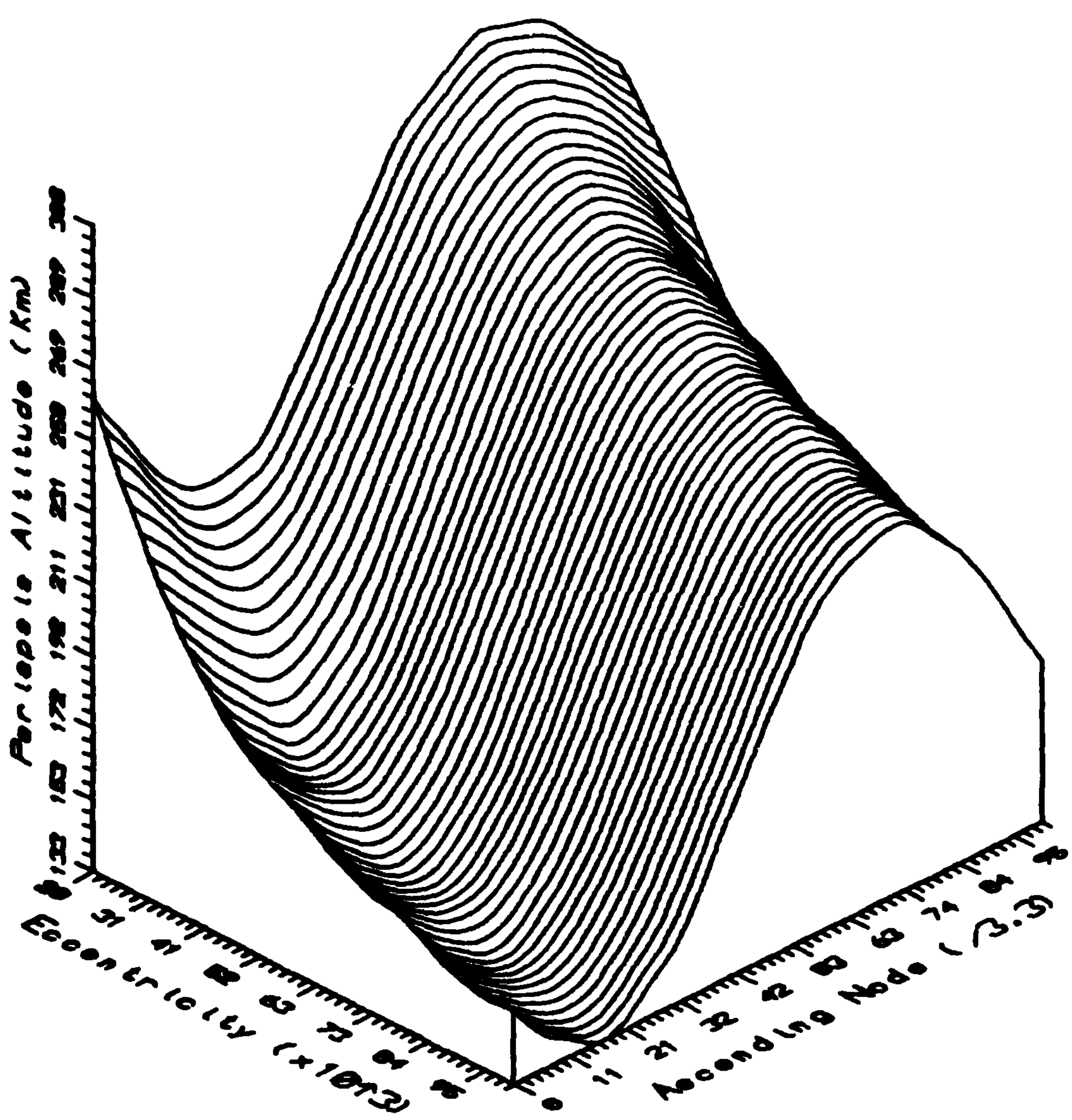

Fig E.15: Periapsis Altitude (i = 7.5 deg)

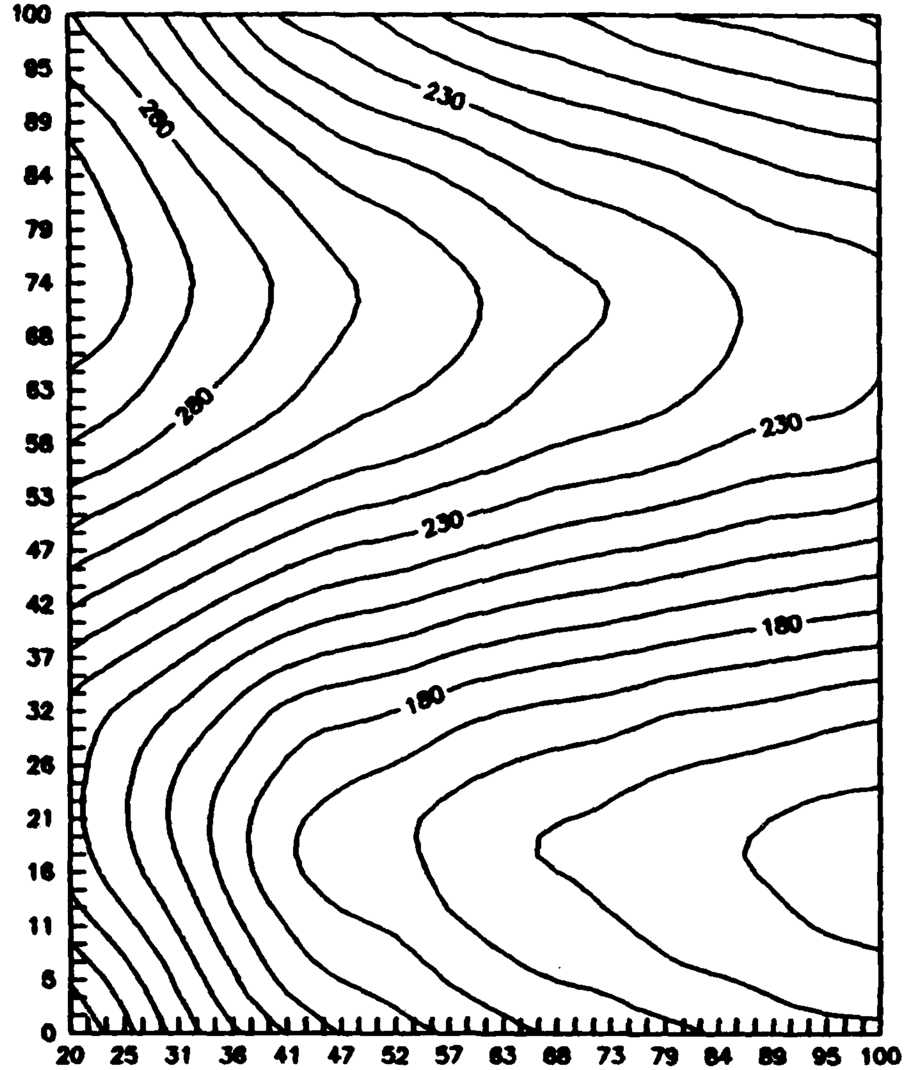

Fig E.16: Periapsis Altitude (i = 7.5 deg)

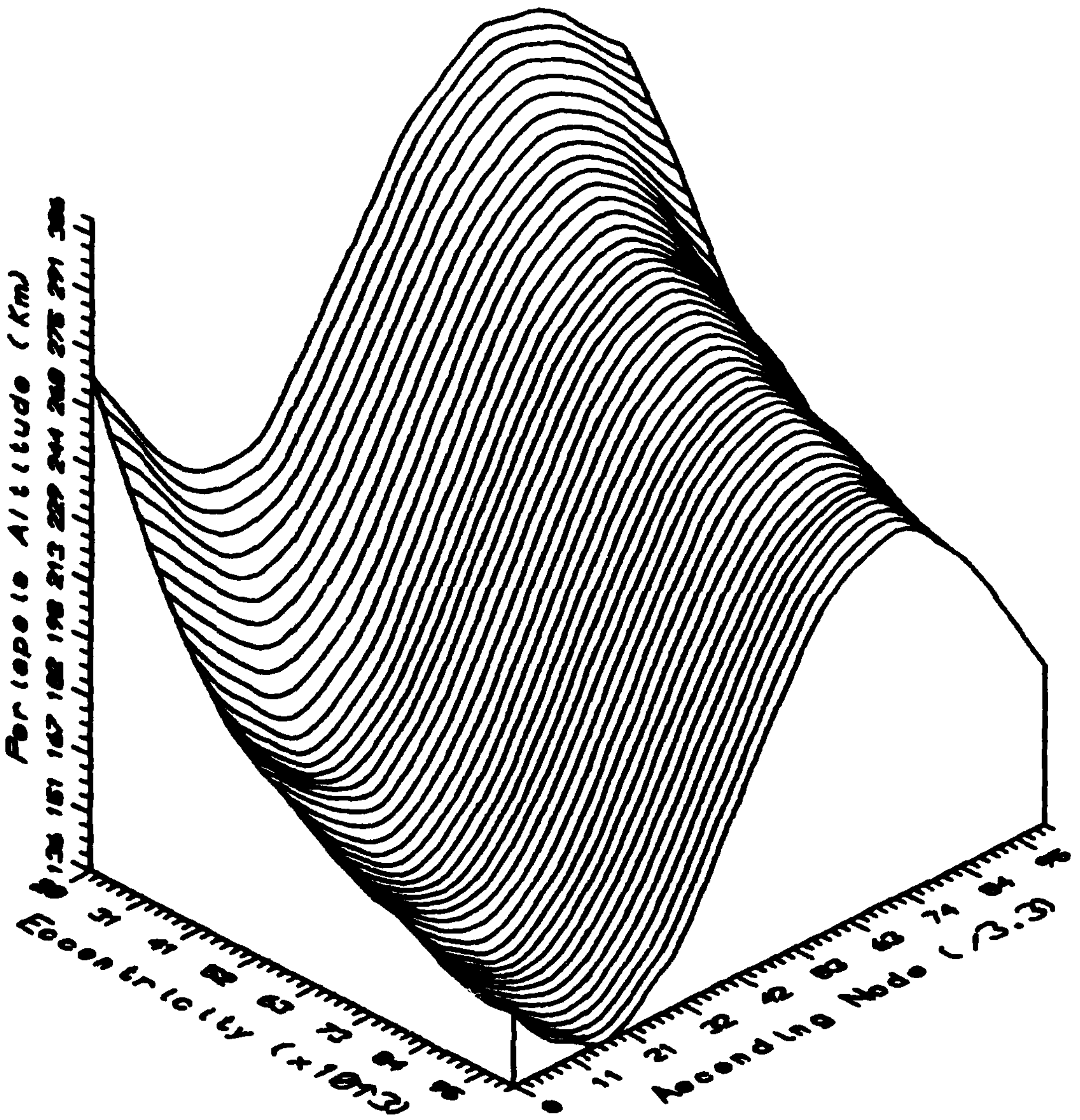

Fig E.17: Periapsis Altitude (ι = 8.5 deg)

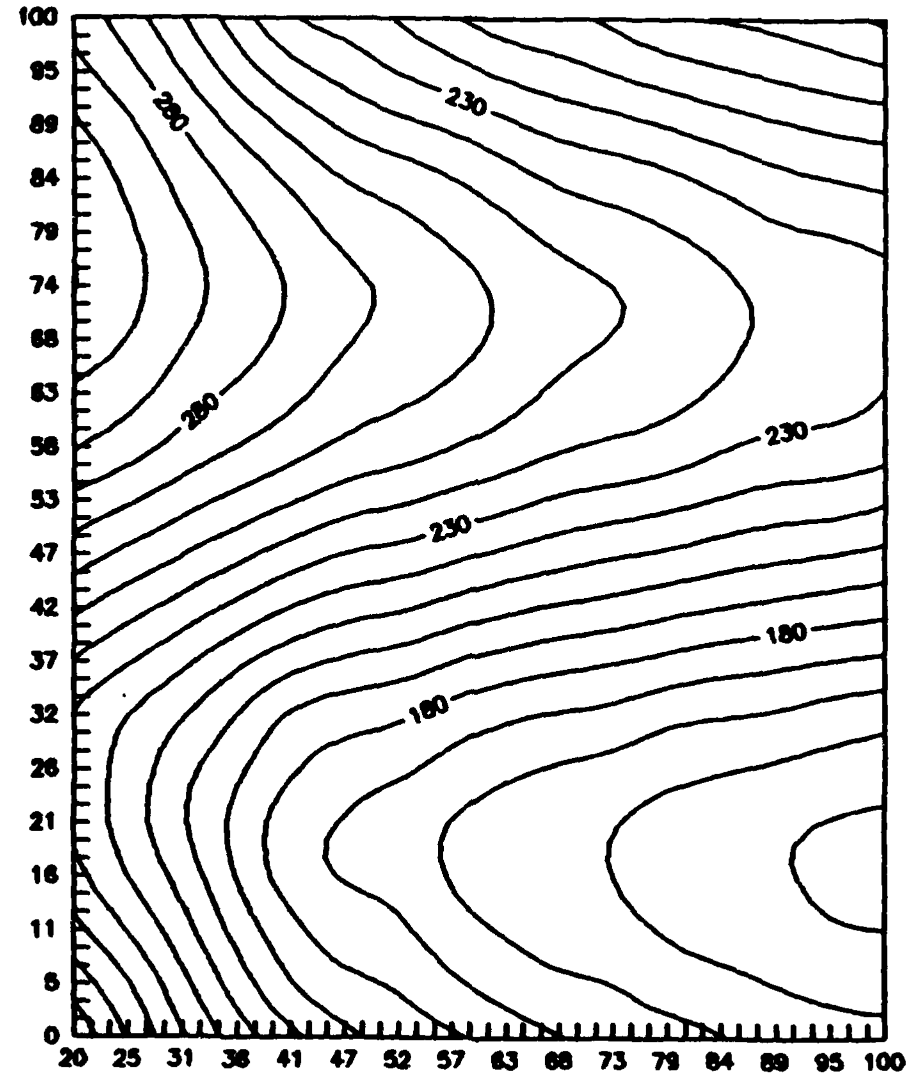

Fig E.18: Periapsis Altitude (i = 8.5 deg)

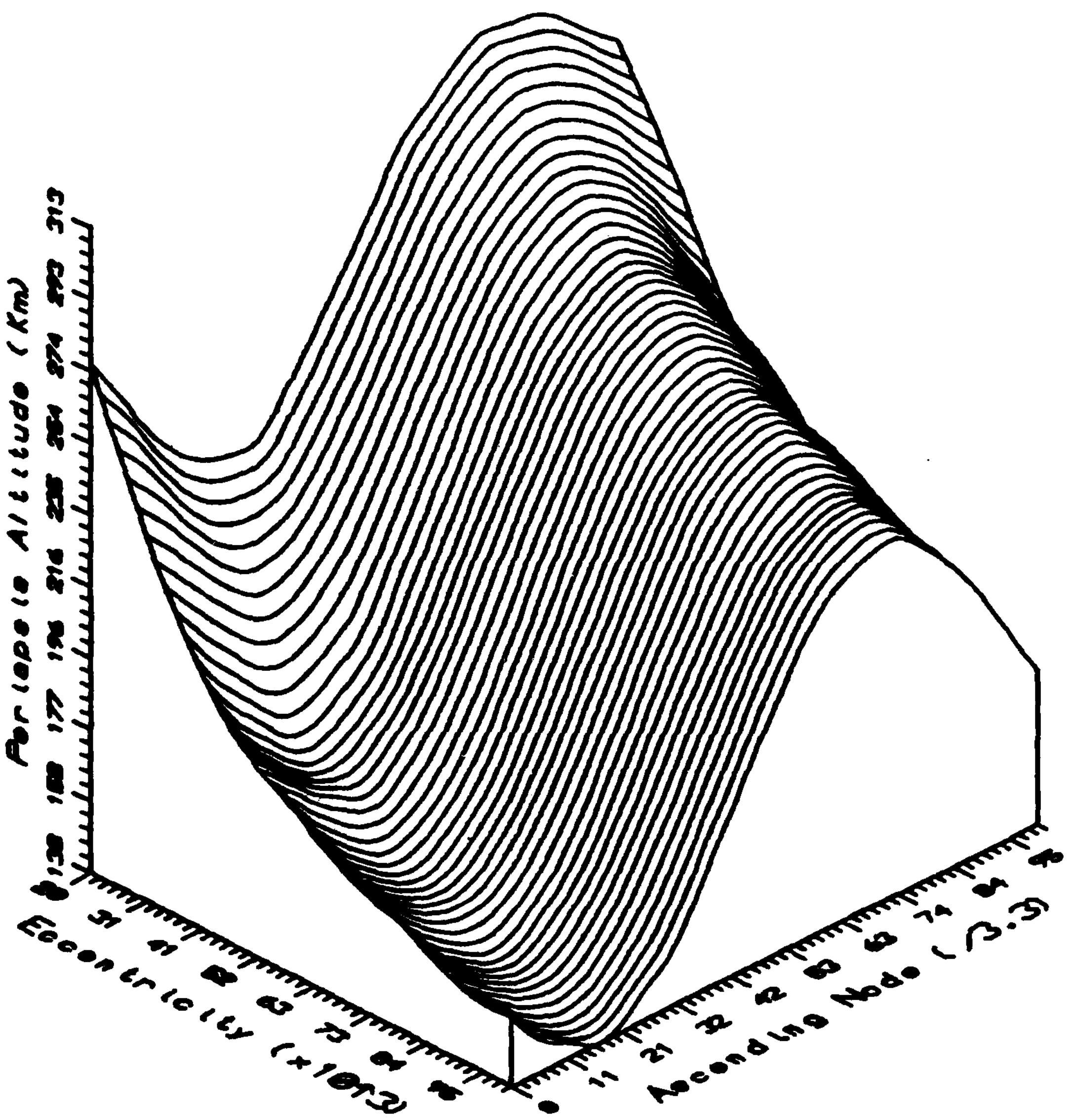

Fig E.19: Periapsis Altitude (i = 9.5 deg)

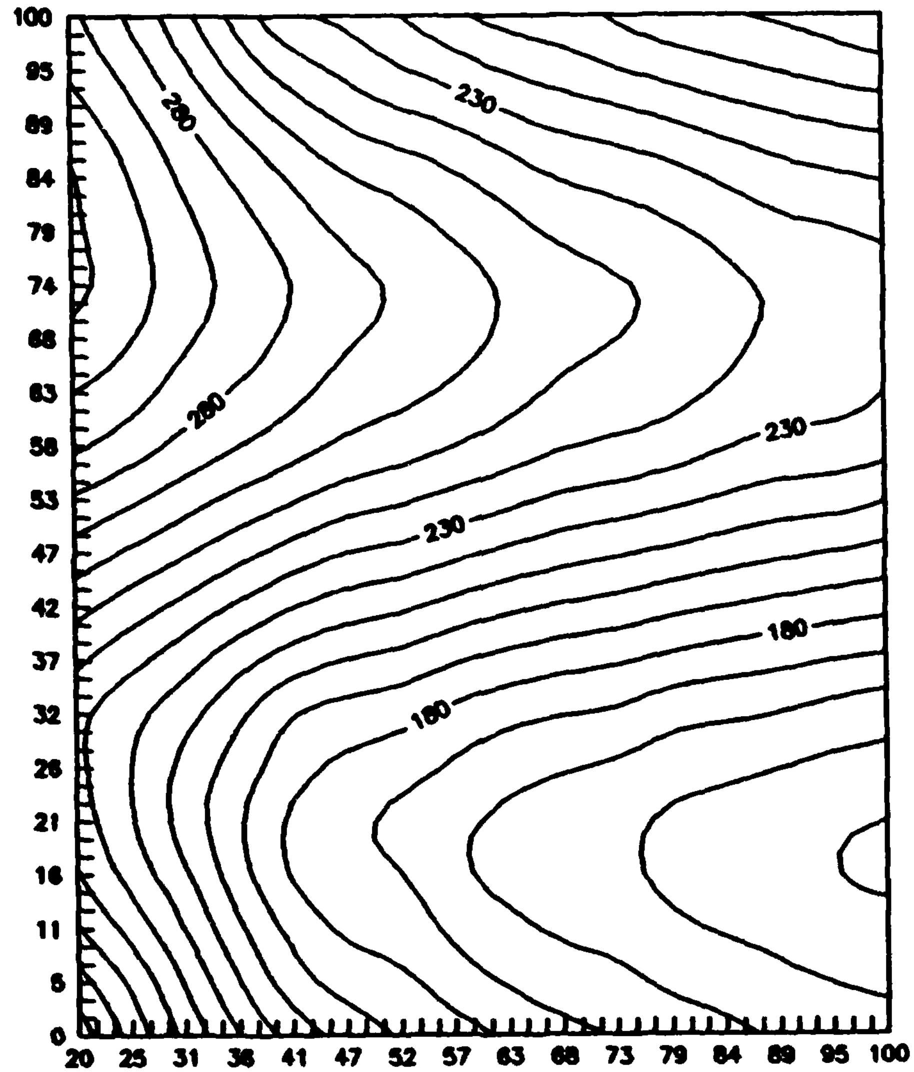

Fig E.20: Periapsis Altitude (i = 9.5 deg)

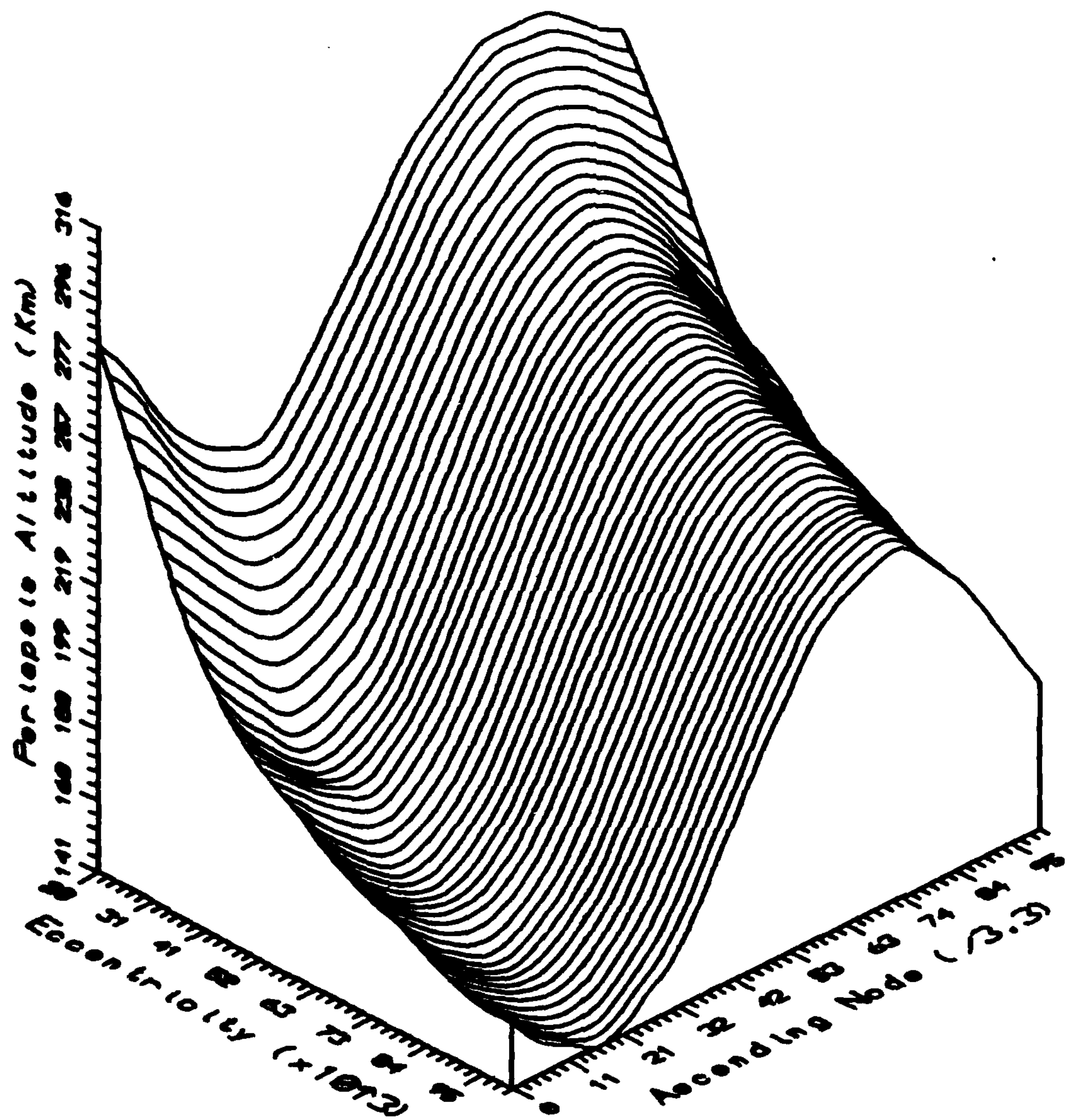

Fig E.21: Periapsis Altitude (i = 10.5 deg)

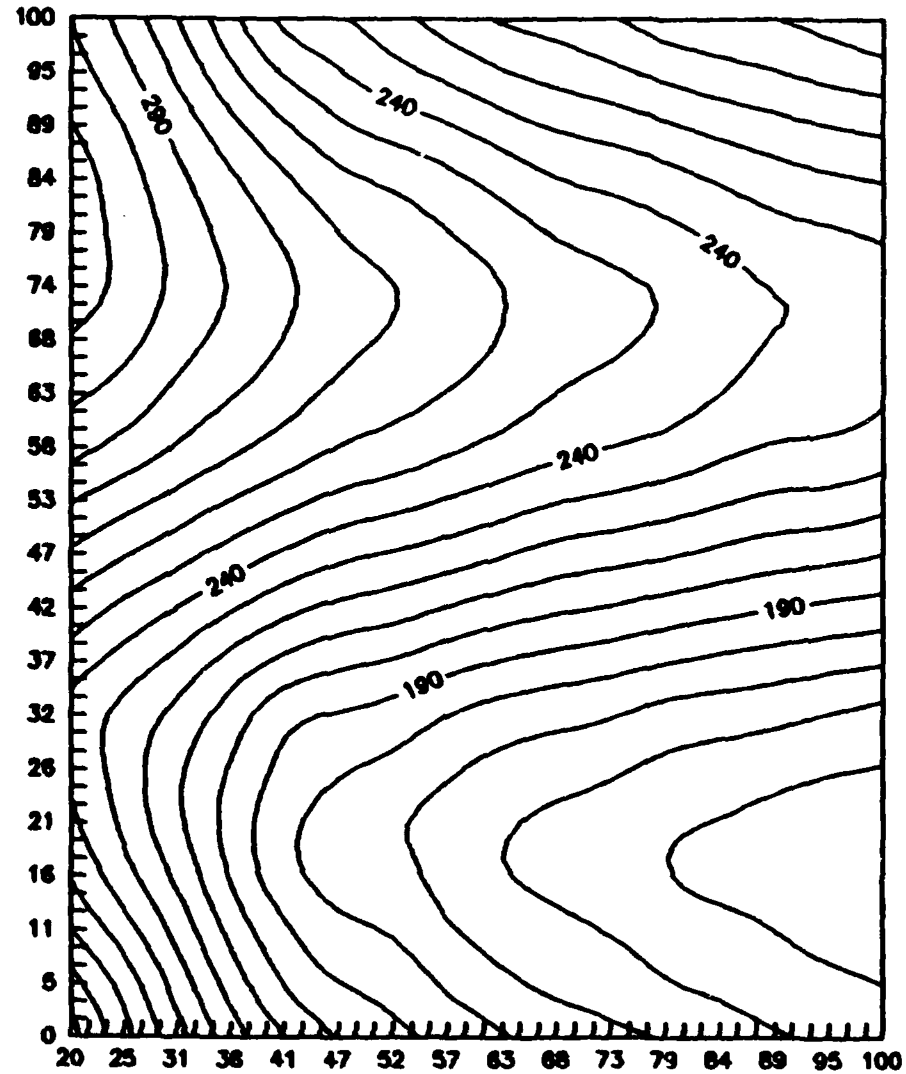

Fig E.22: Periapsis Altitude (i = 10.5 deg)

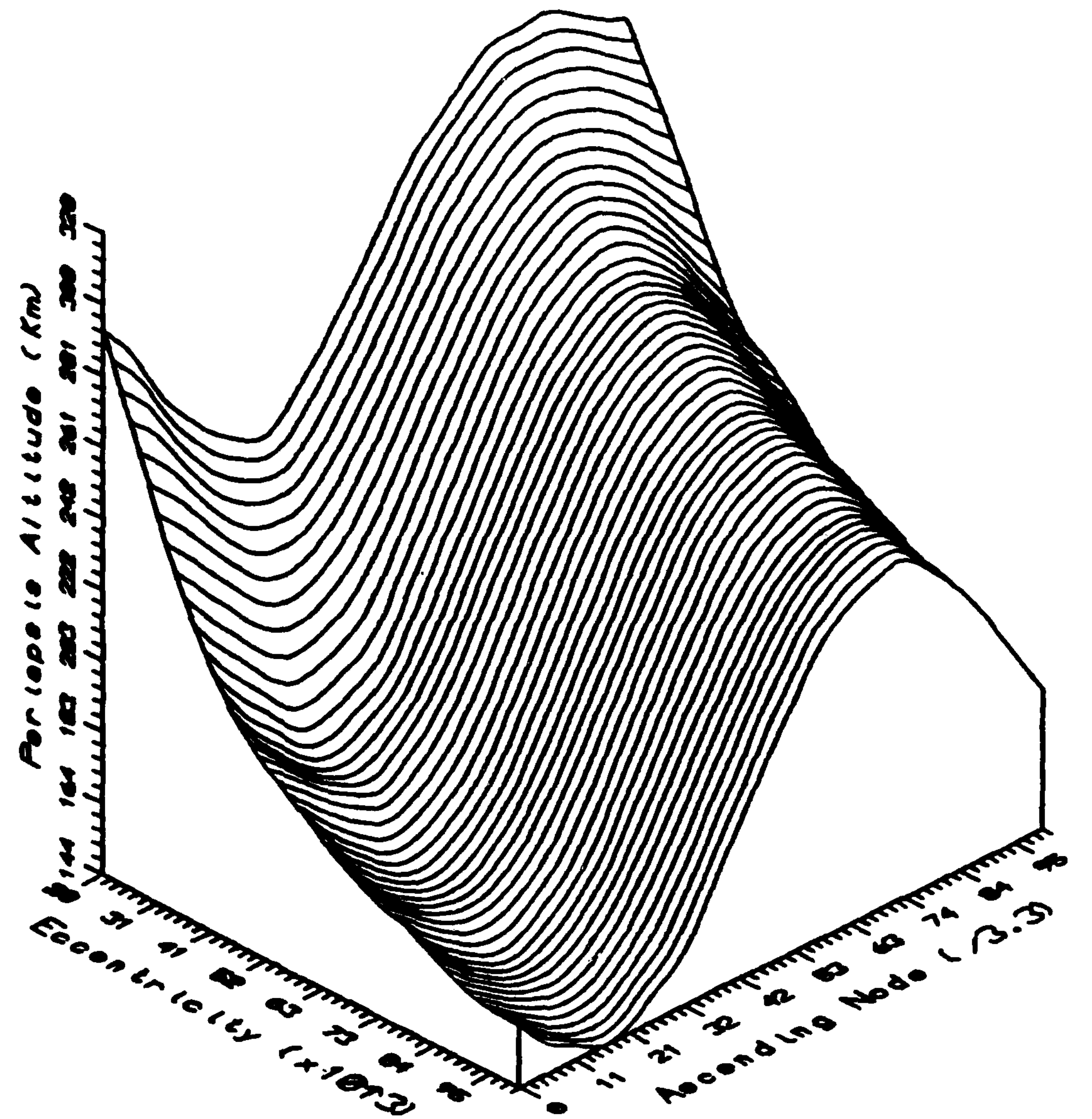

Fig E.23: Periapsis Altitude (i = 11.5 deg)

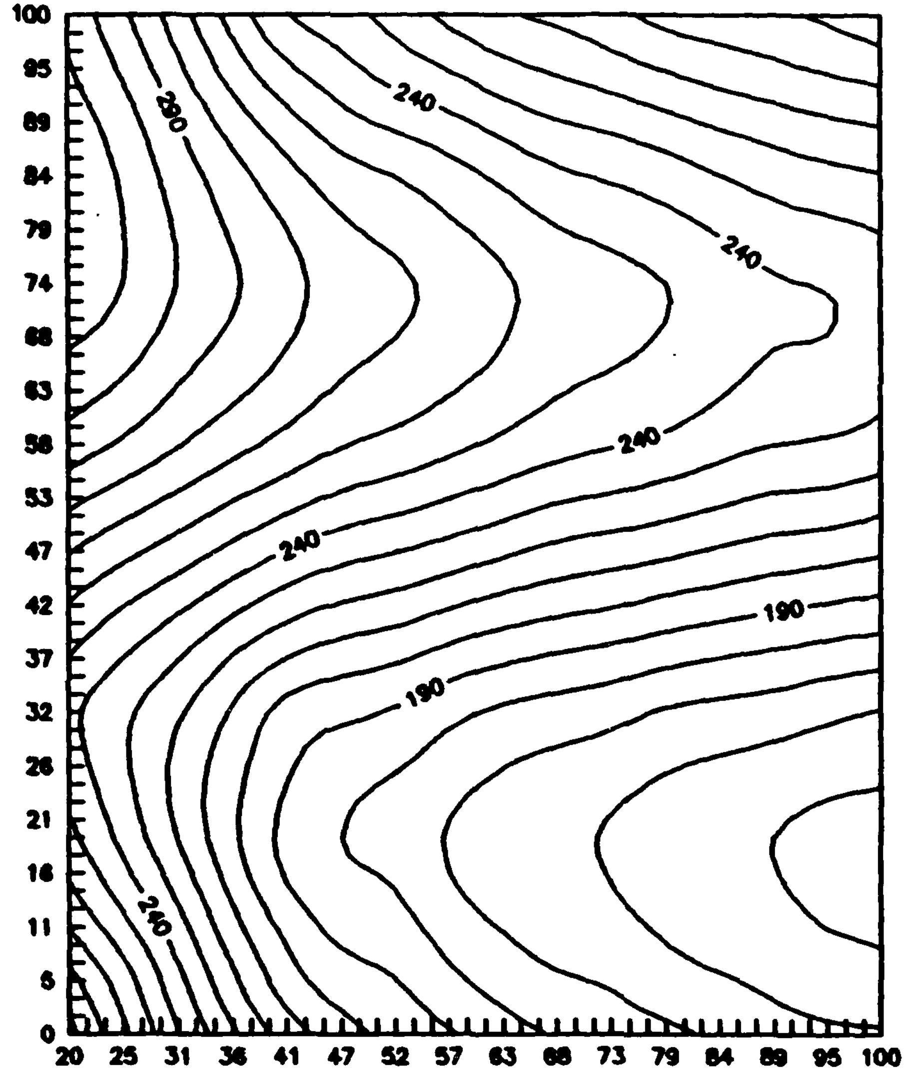

Fig E.24: Periapsis Altitude (i = 11.5 deg)

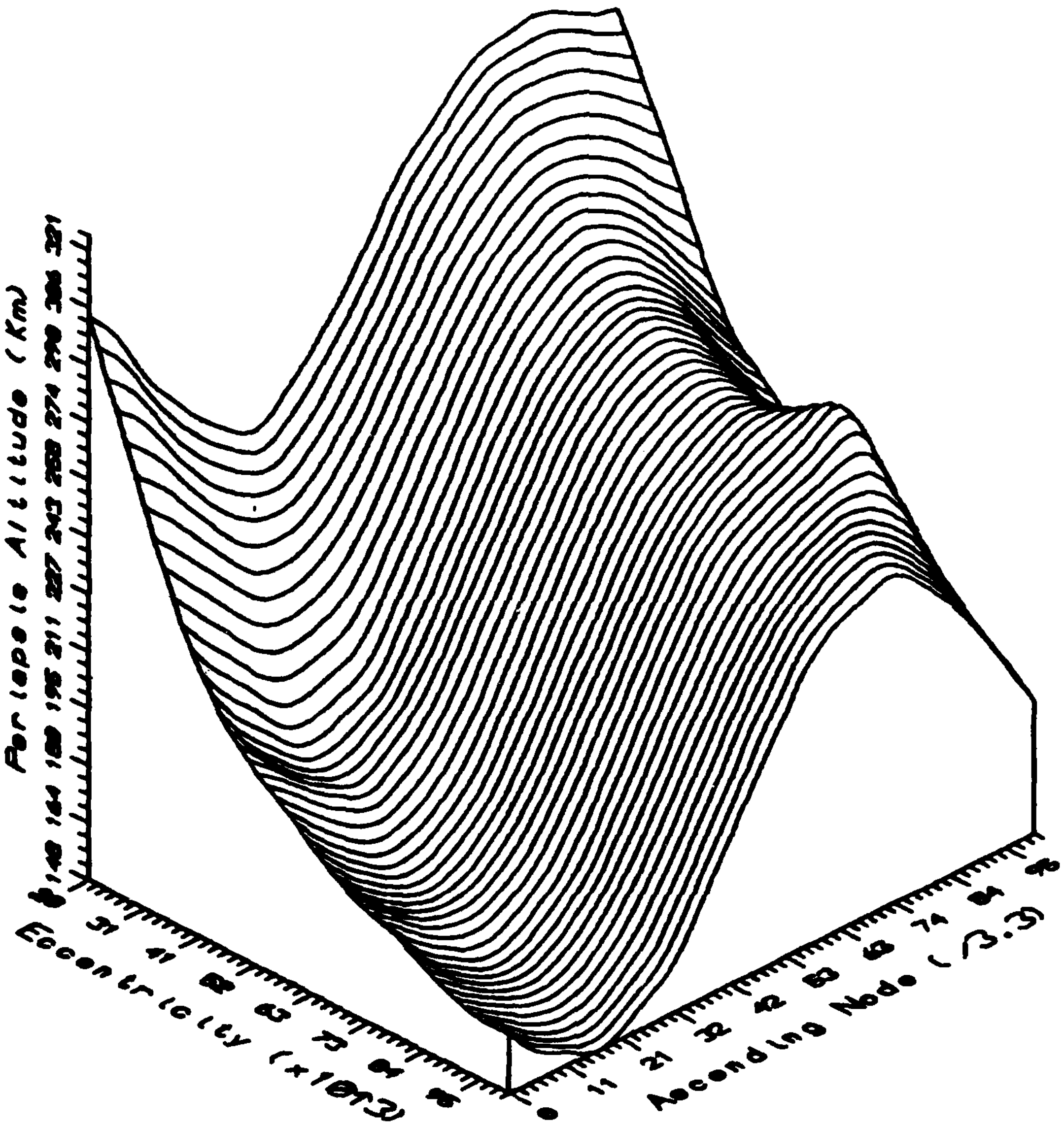

Fig E.25: Periapsis Altitude (i = 12.5 deg)

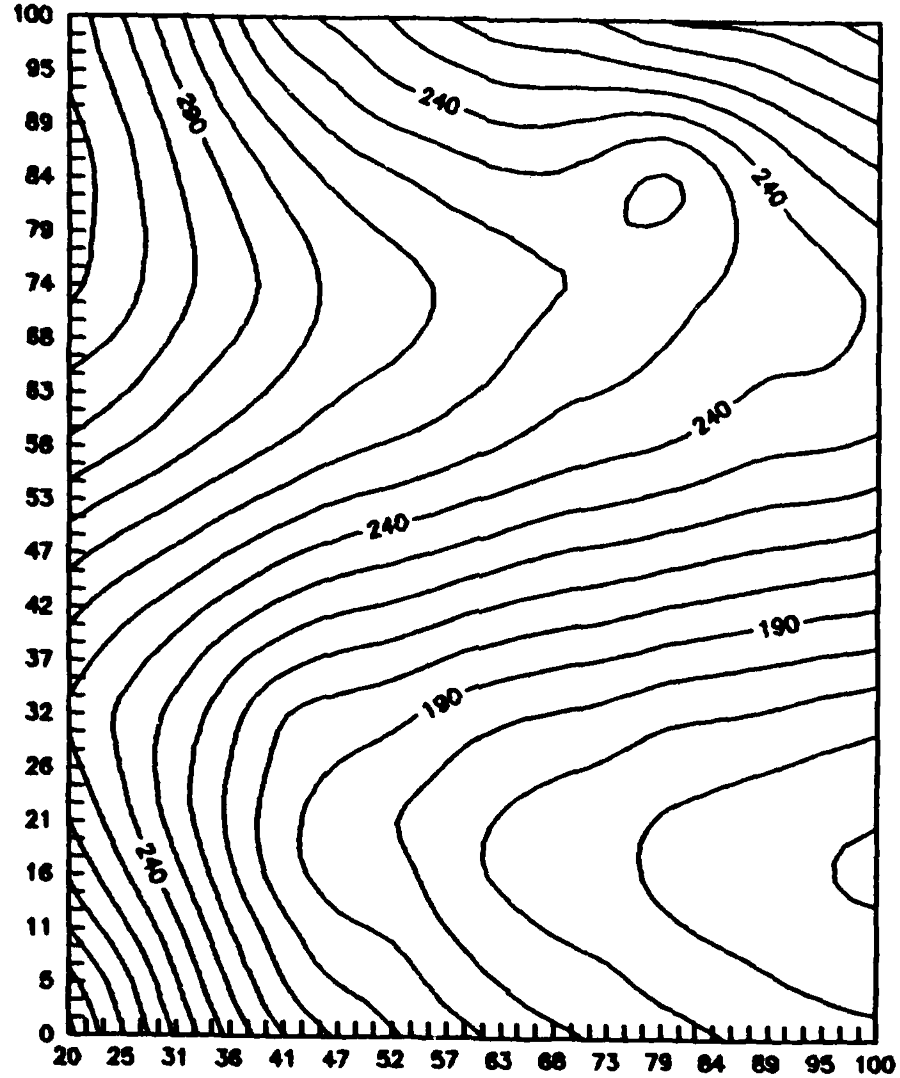

Fig E.26: Periapsis Altitude (i = 12.5 deg)

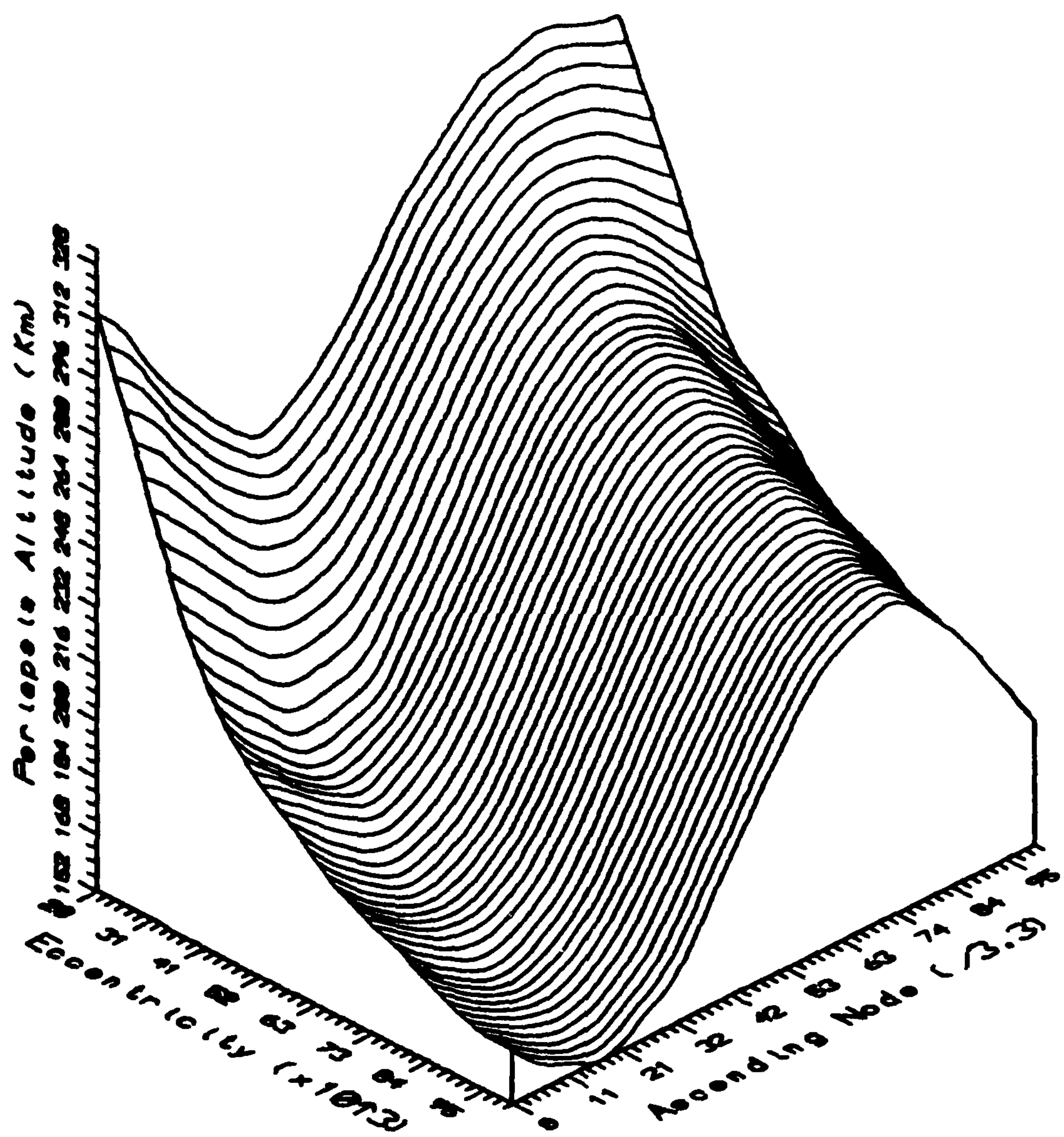

Fig E.27: Periapsis Altitude (i = 13.5 deg)

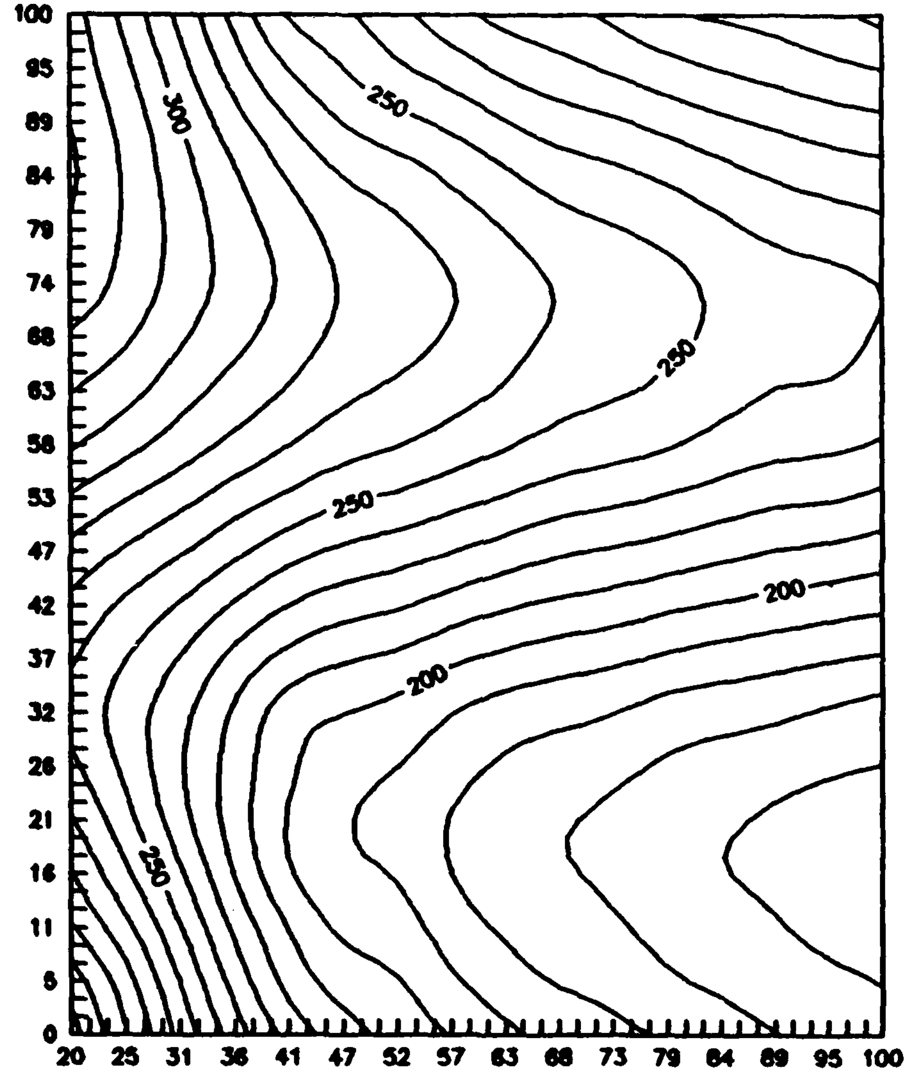

Fig E.28: Periapsis Altitude (i = 13.5 deg)

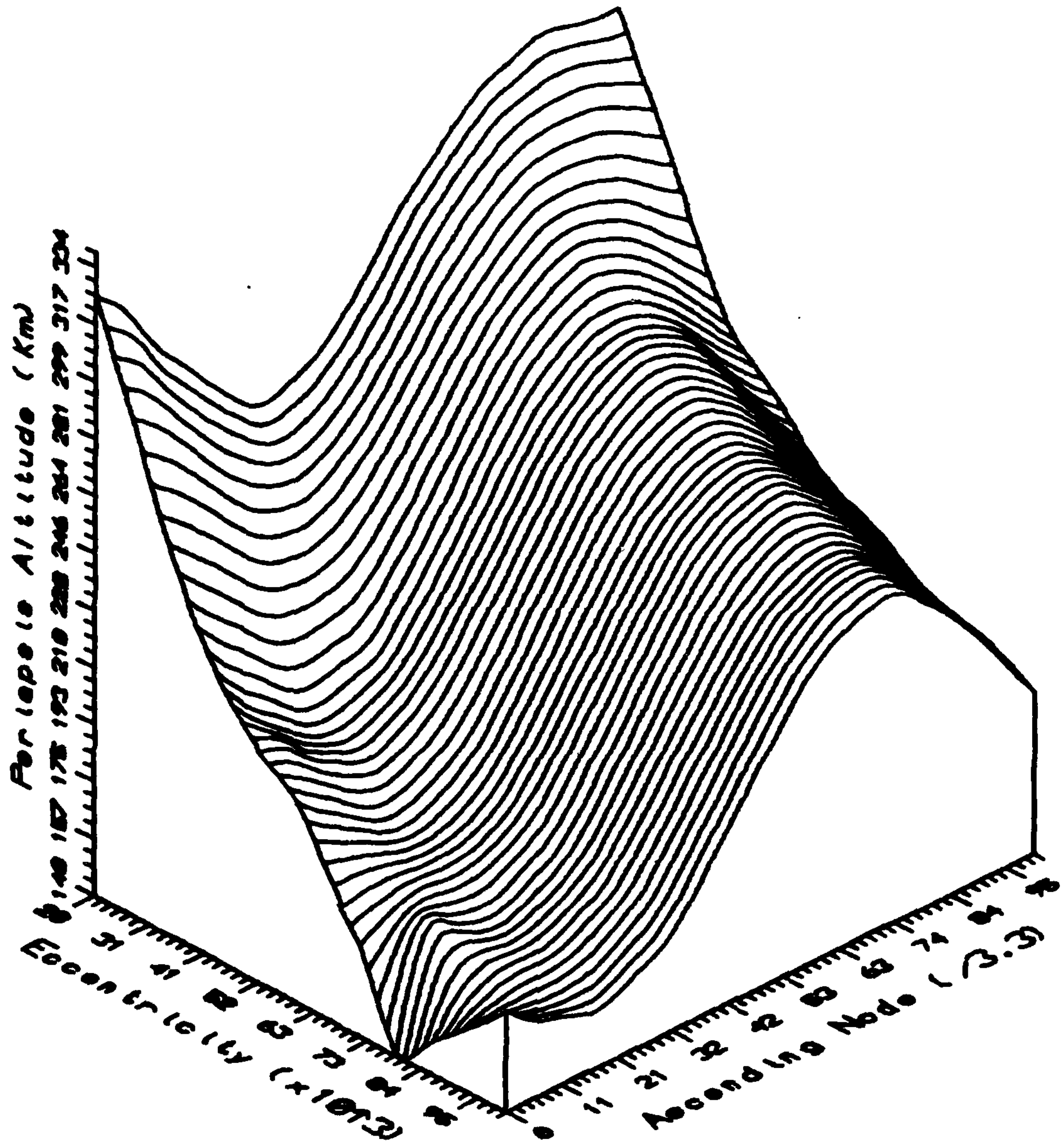

Fig E.29: Periapsis Altitude (i = 14.5 deg)

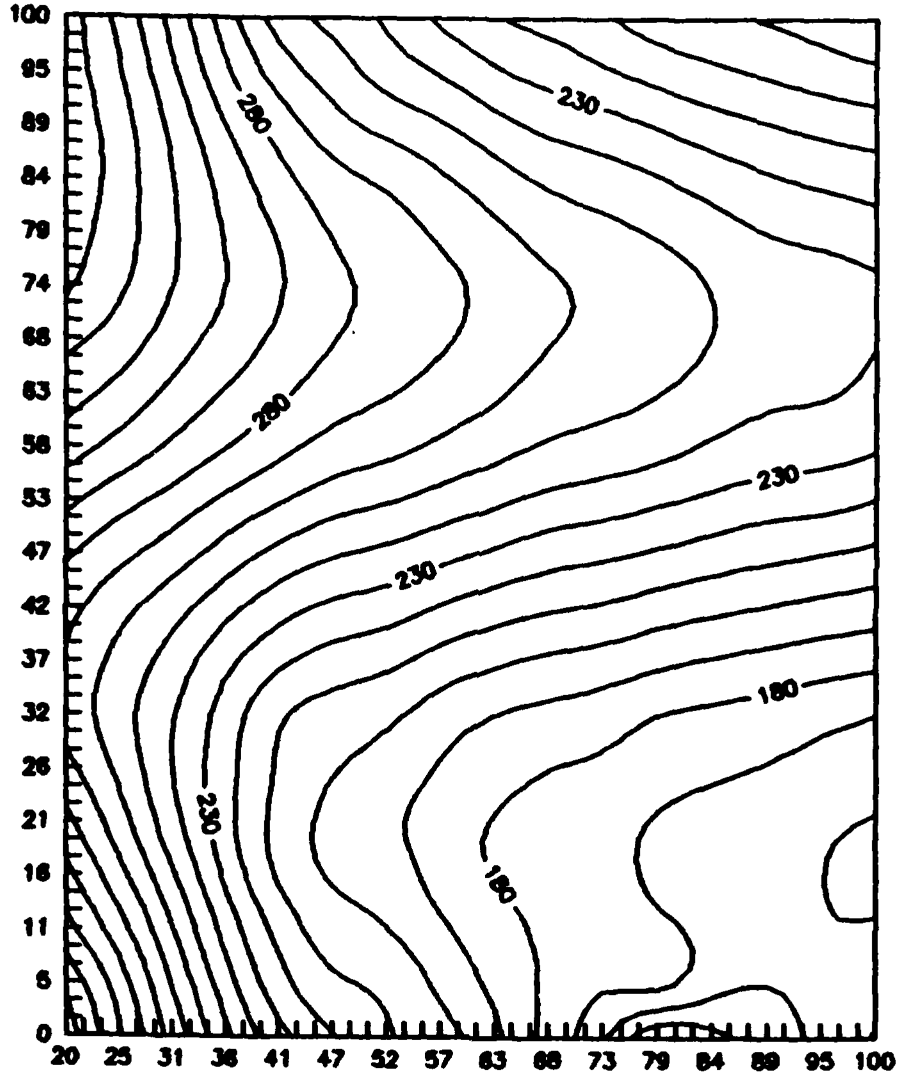

Fig E.30: Periapsis Altitude (i = 14.5 deg)

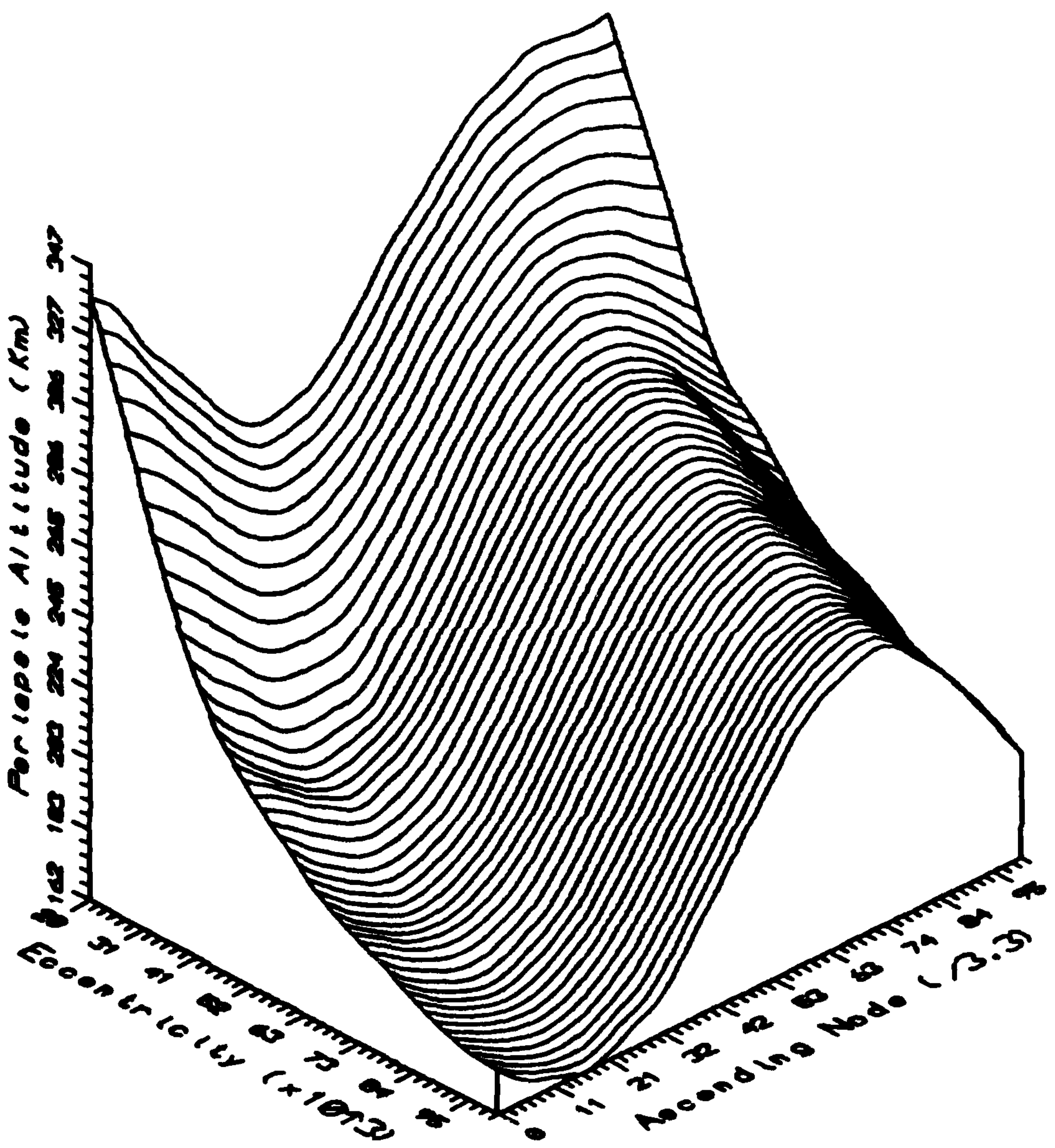

Fig E.31. Periapsis Altitude (i = 15.5 deg)

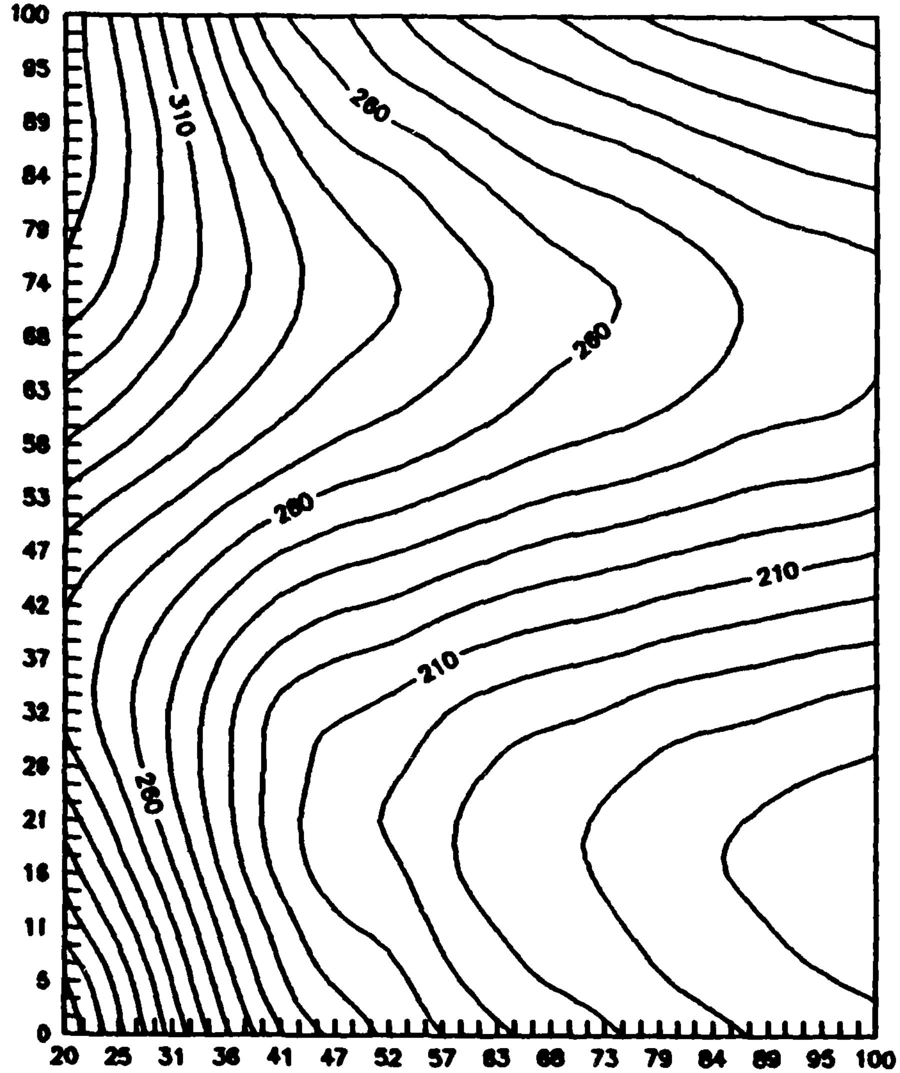

Fig E.32: Periapsis Altitude (i = 15.5 deg)

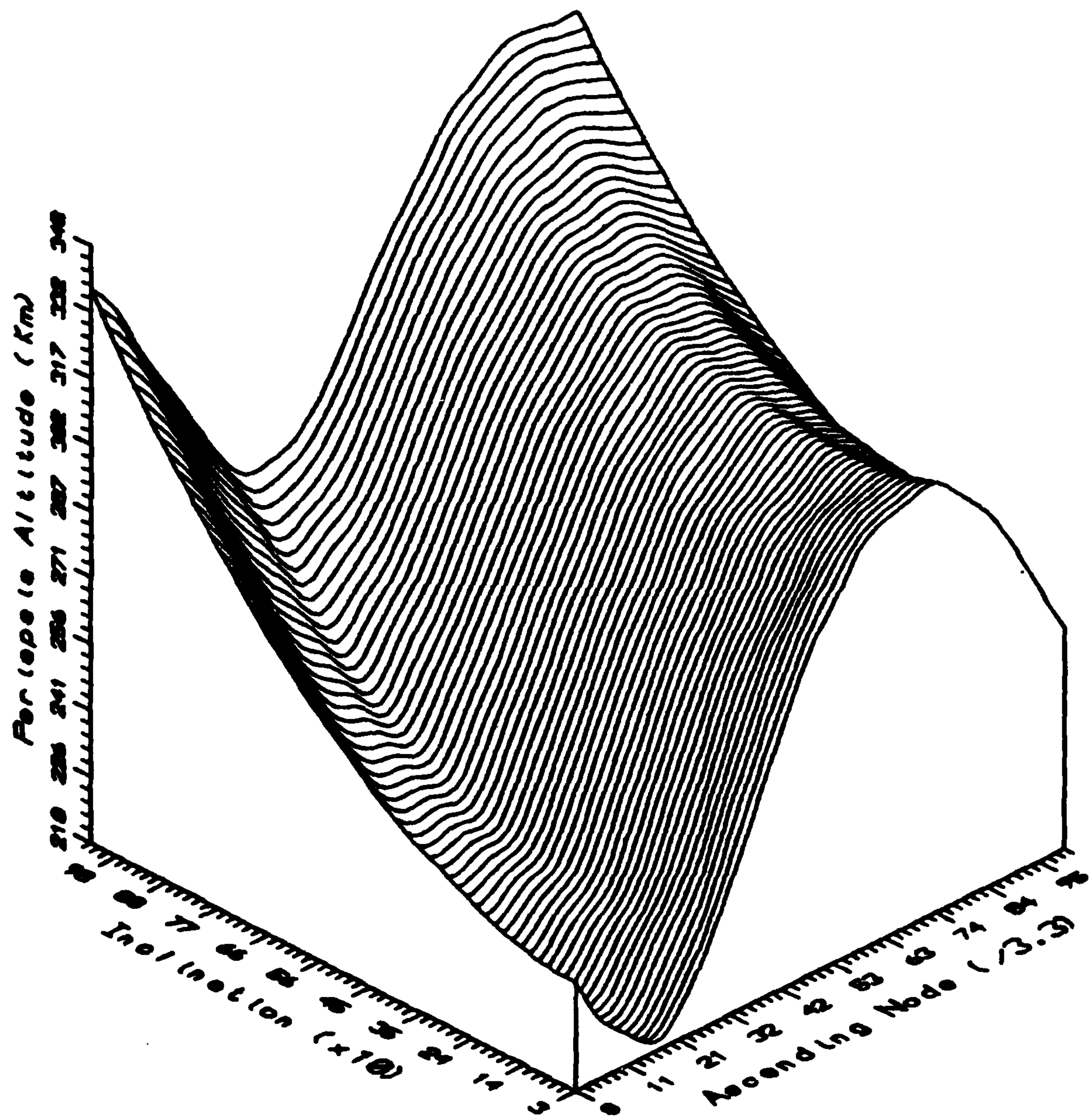

Fig E.33: Periapsis Altitude (e = 0.02)

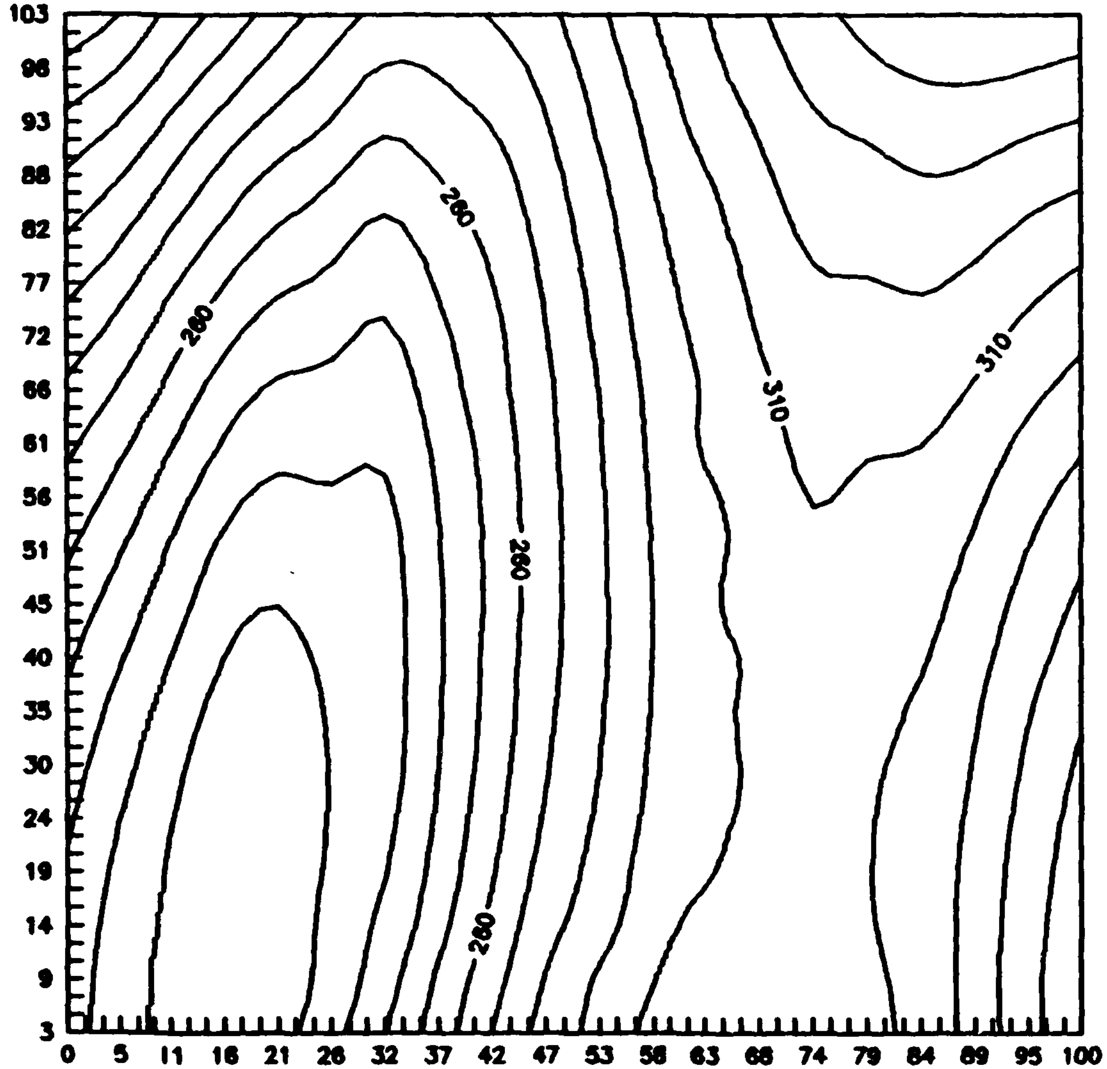

Fig E.34: Periapsis Contour (e = 0.02)

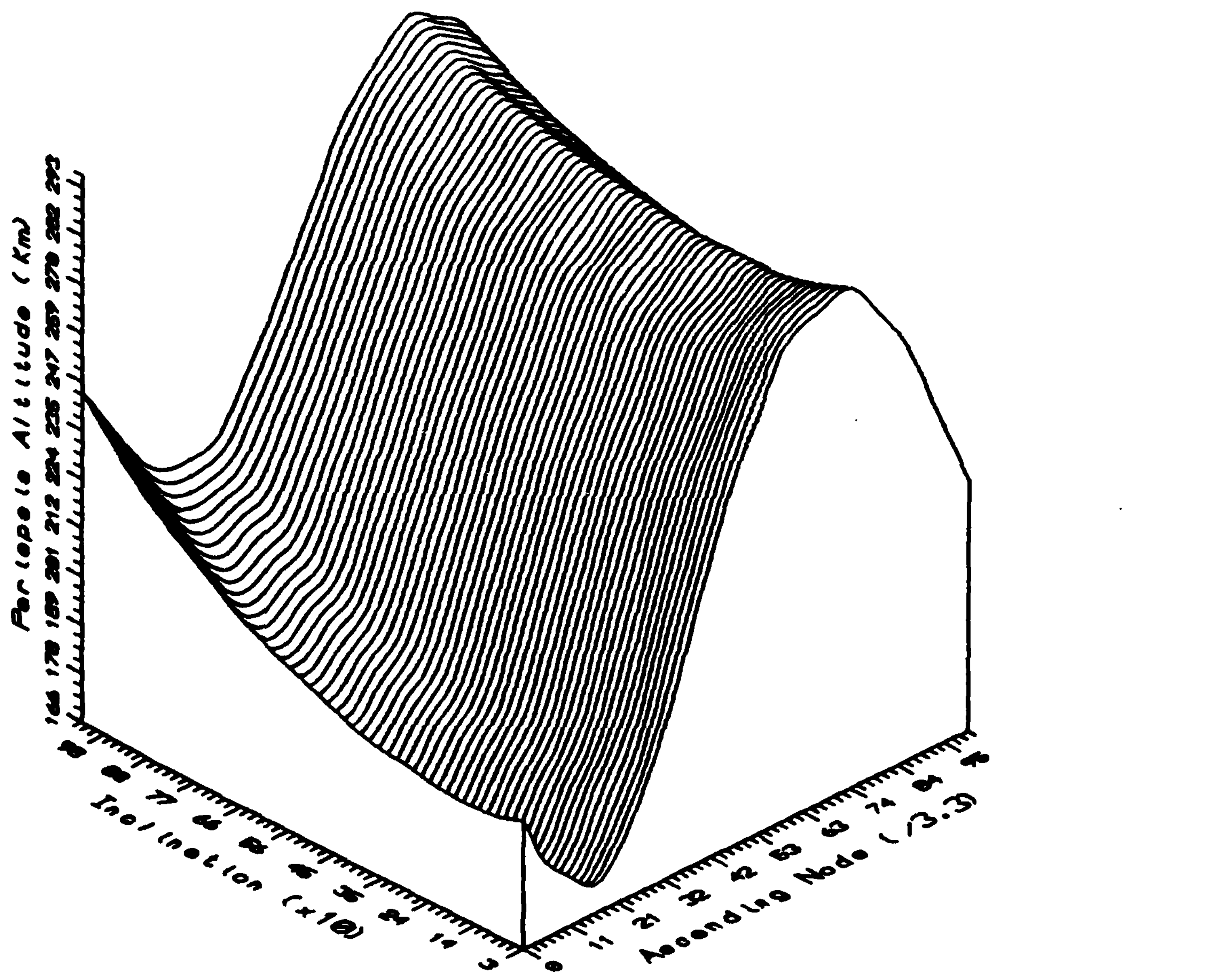

Fig E.35: Periapsis Altitude (e = 0.04)

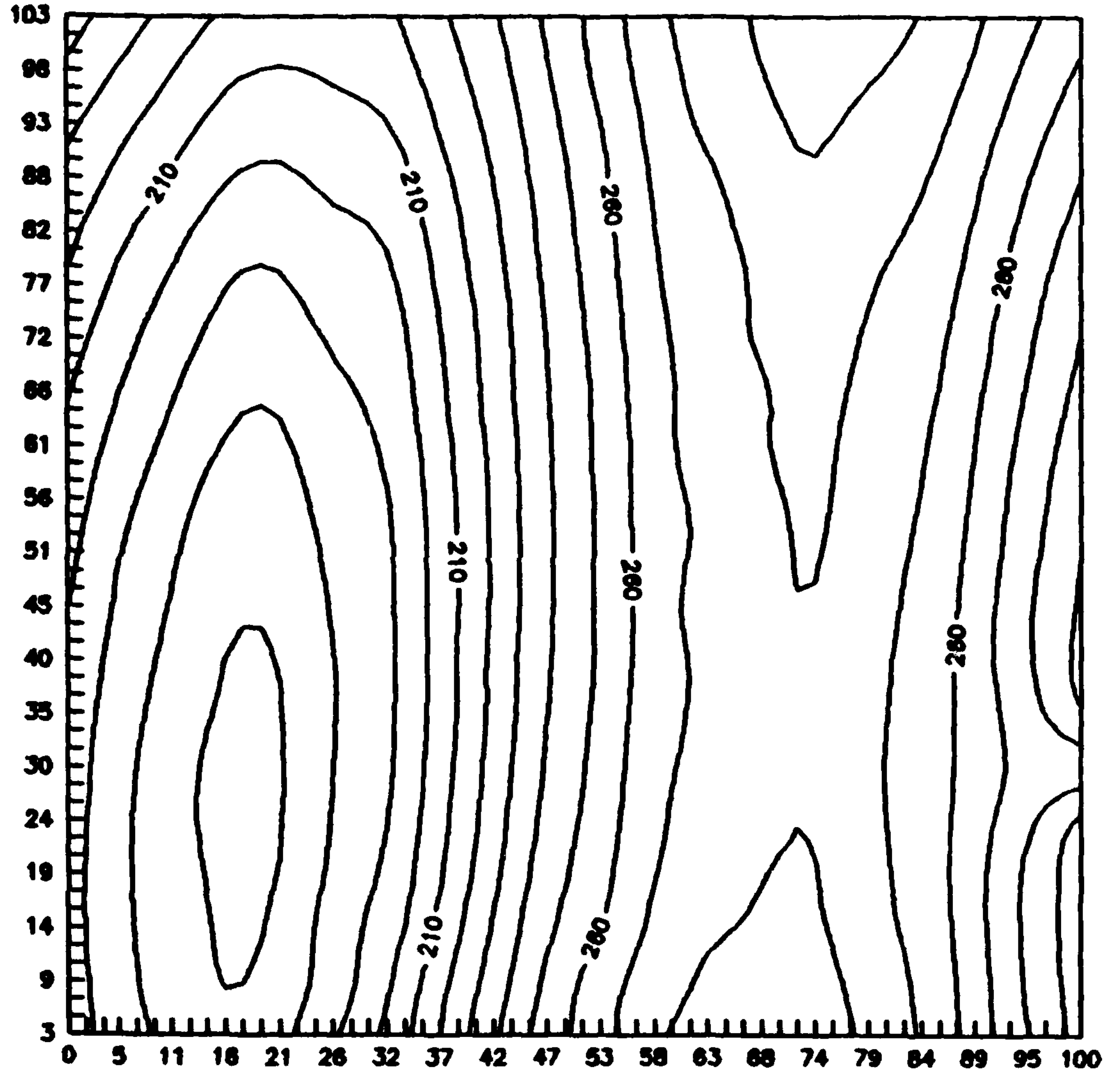

Fig E.36: Periapsis Contour (e = 0.04)

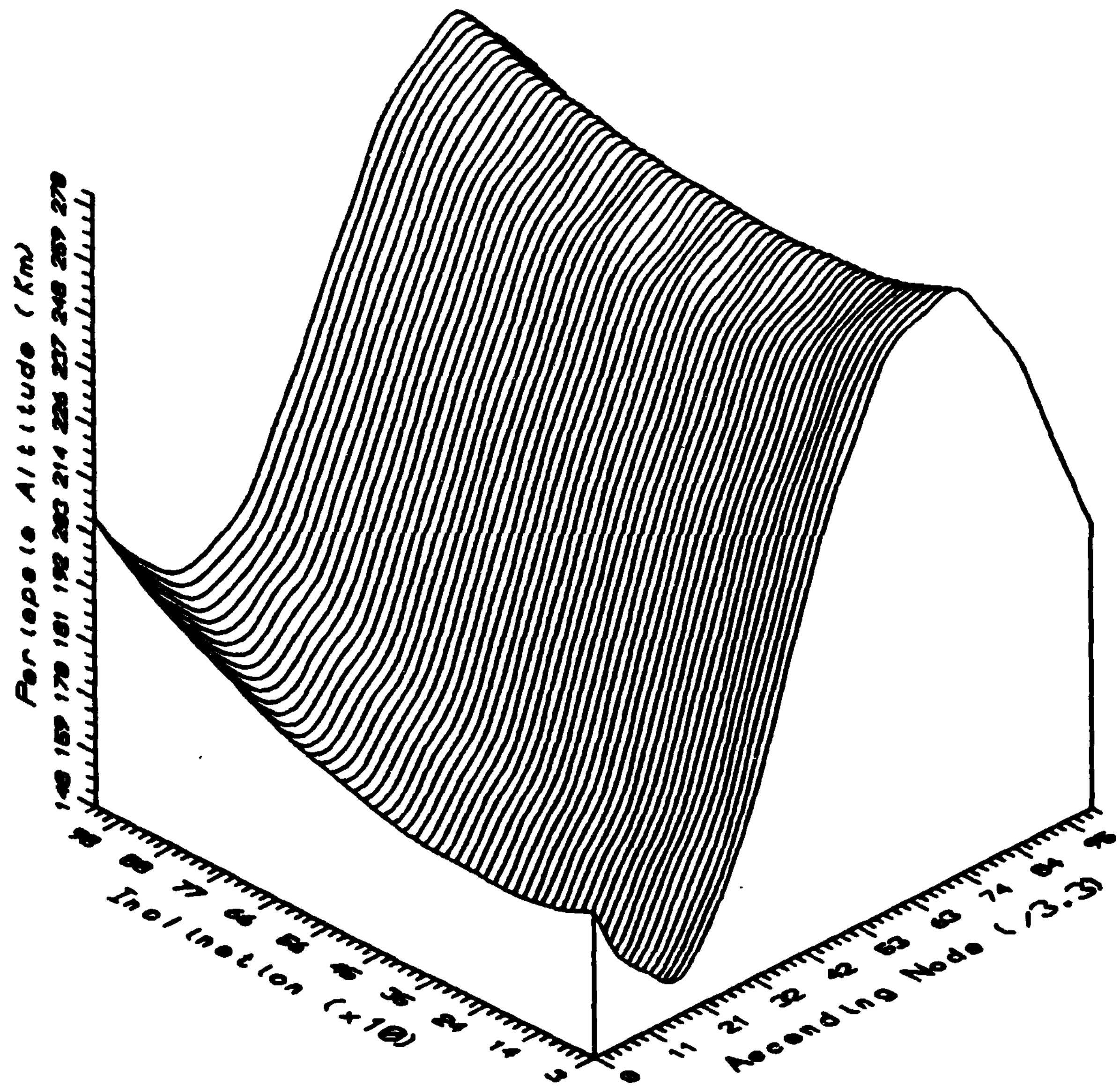

Fig E.37: Periapsis Altitude (e = 0.06)

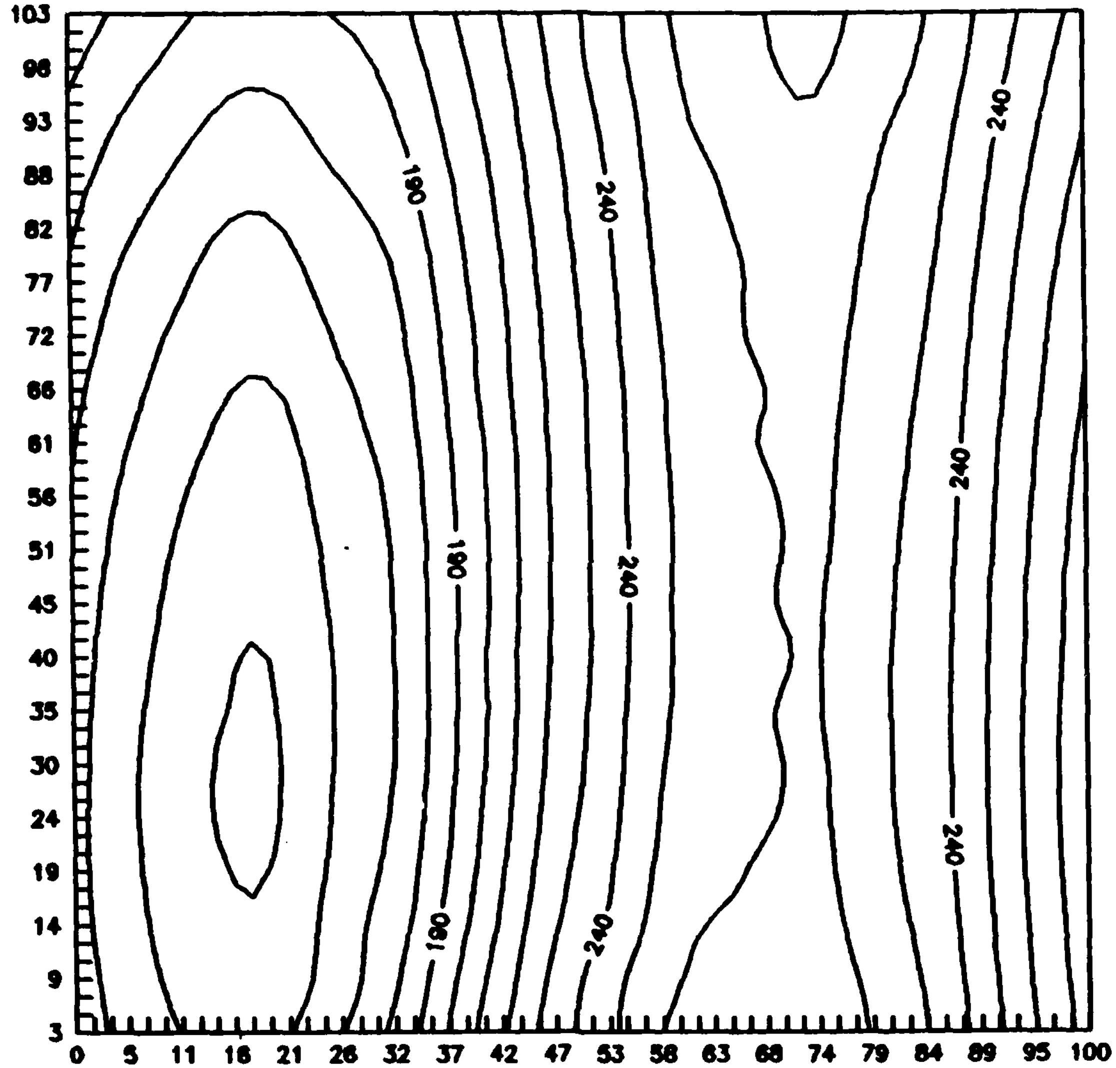

Fig E.38: Periapsis Contour (e = 0.06)

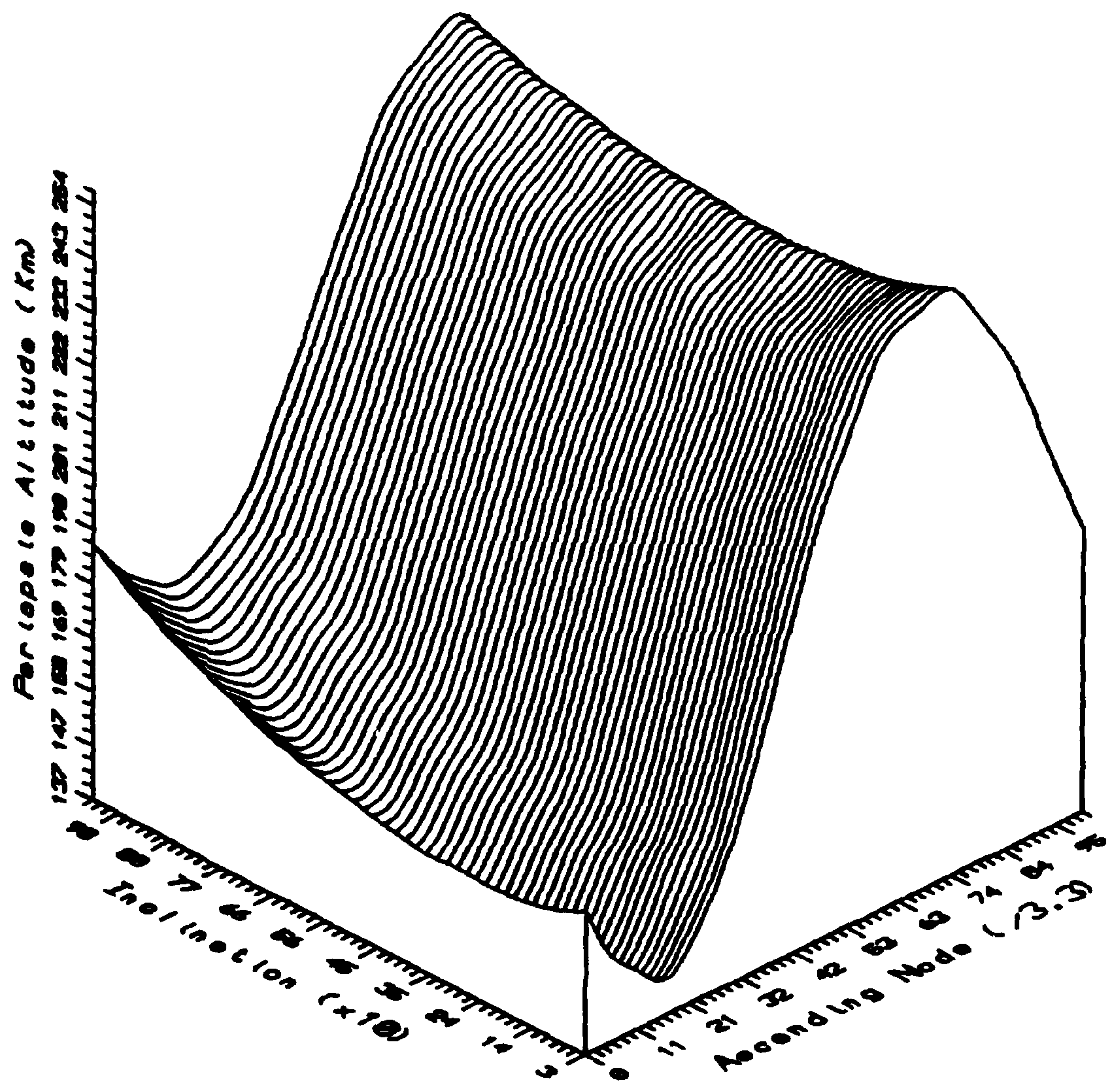

Fig E.39: Perlapsle Altitude (e = 0.08)

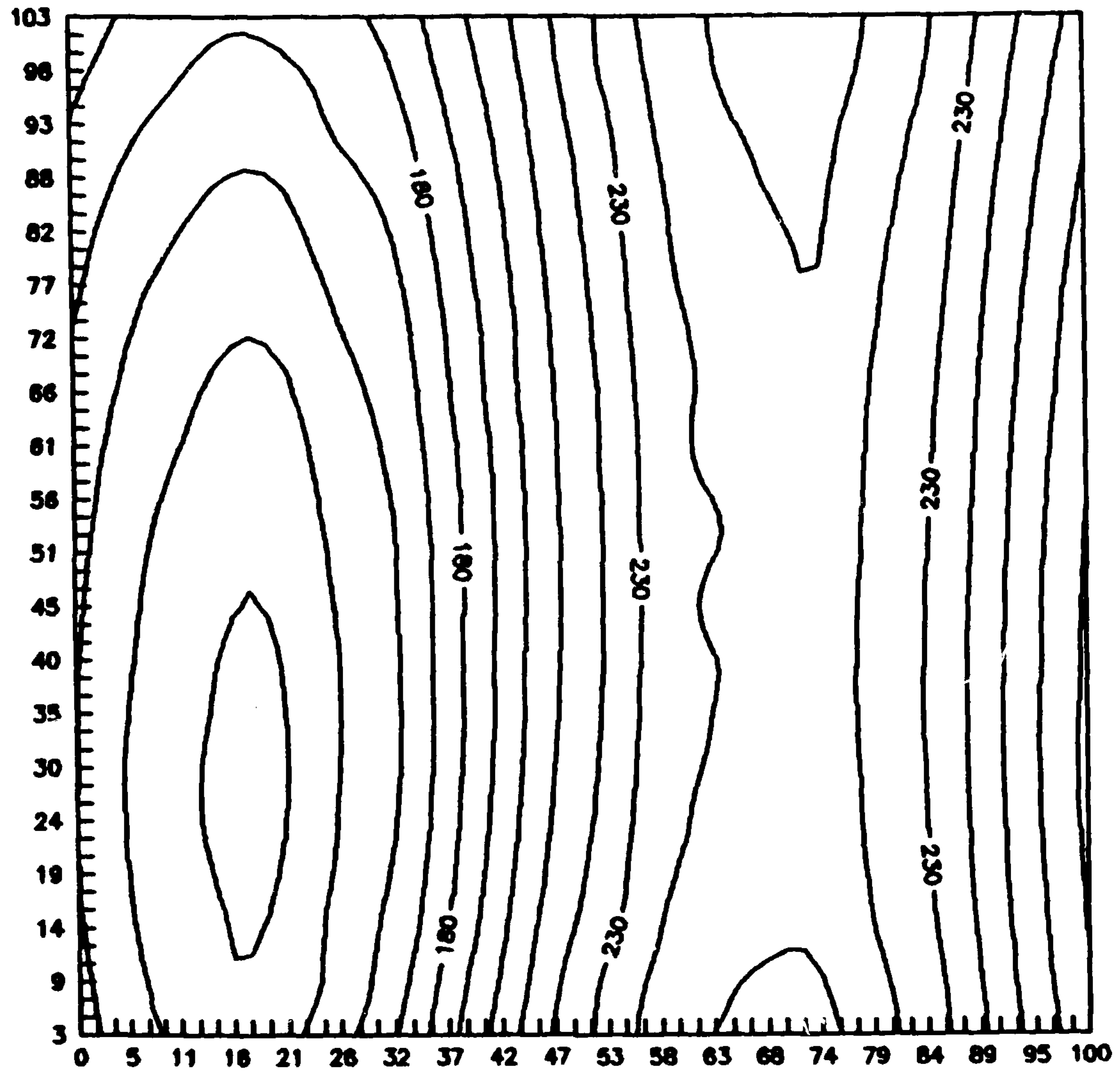

Fig E.40: Periapsis Contour (e = 0.08)

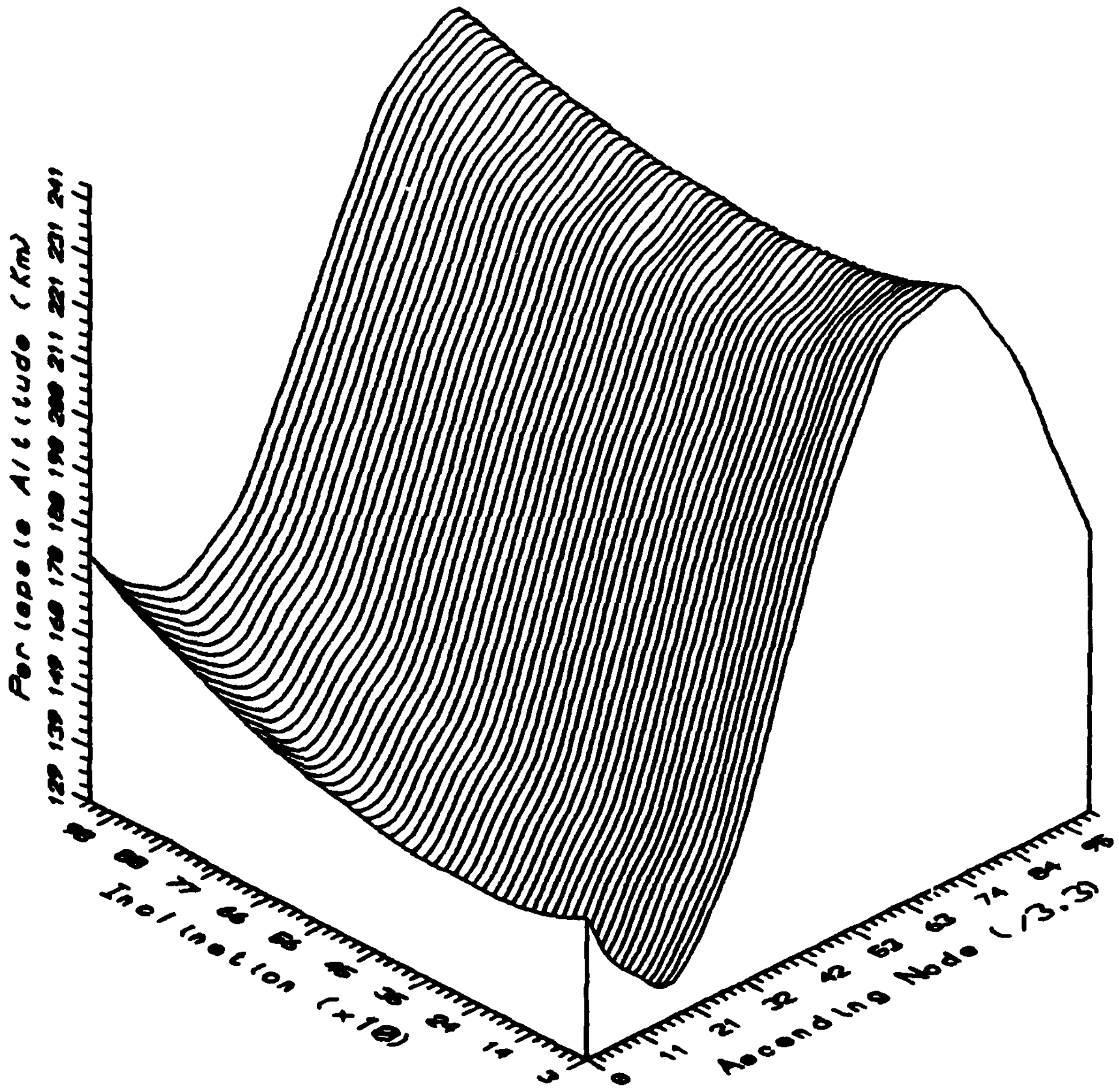

Fig E.41: Periapsis Altitude (e = 0.10)

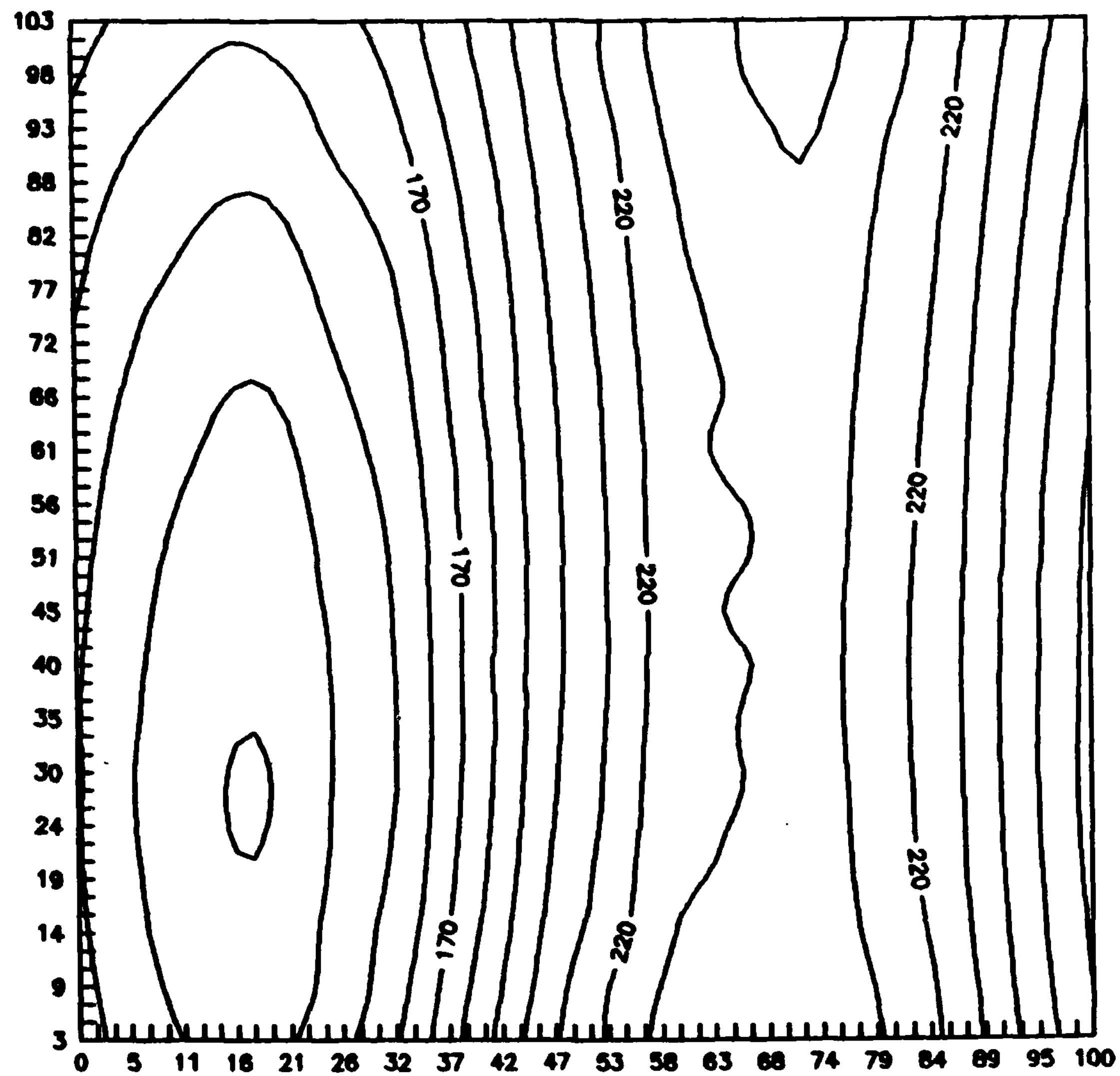

Fig E.42: Periapsis Contour (e = 0.10)

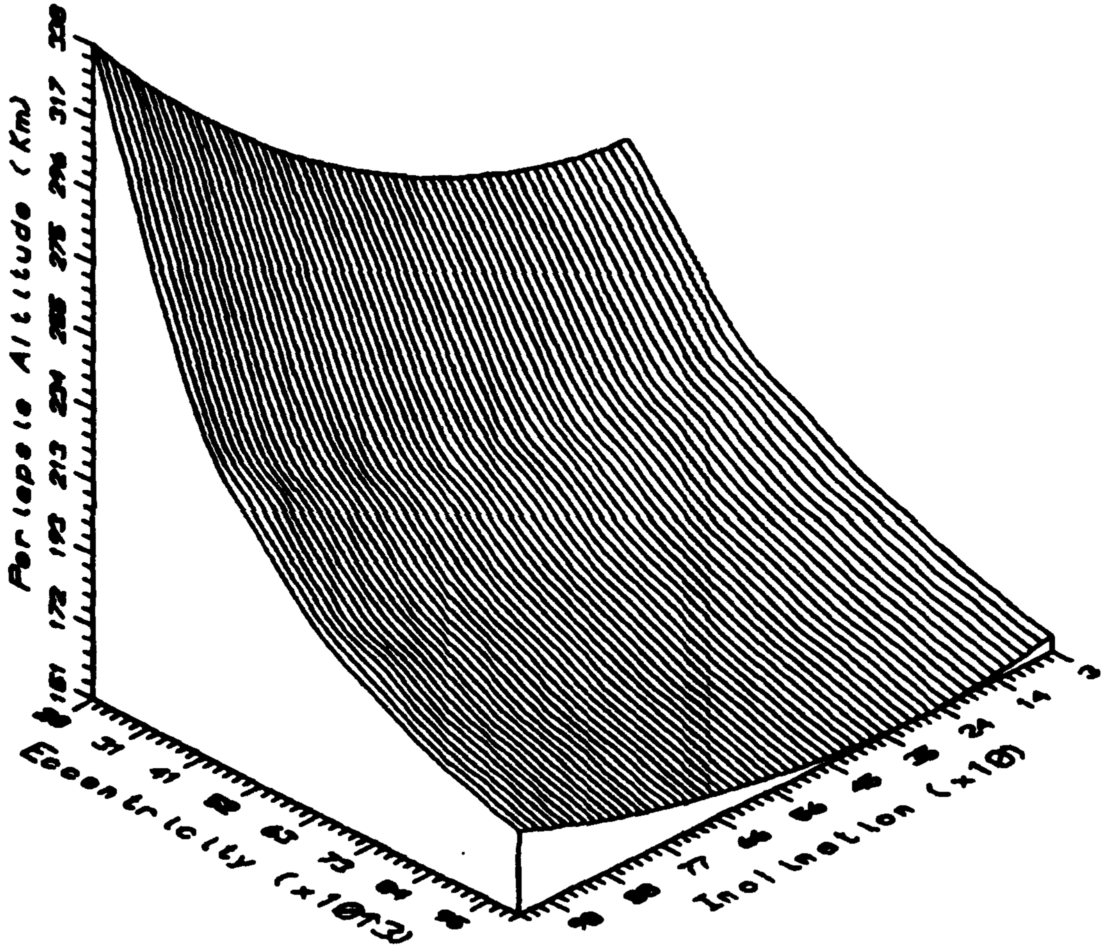

Fig E.43: Periapsis Altitude (= 0 deg)

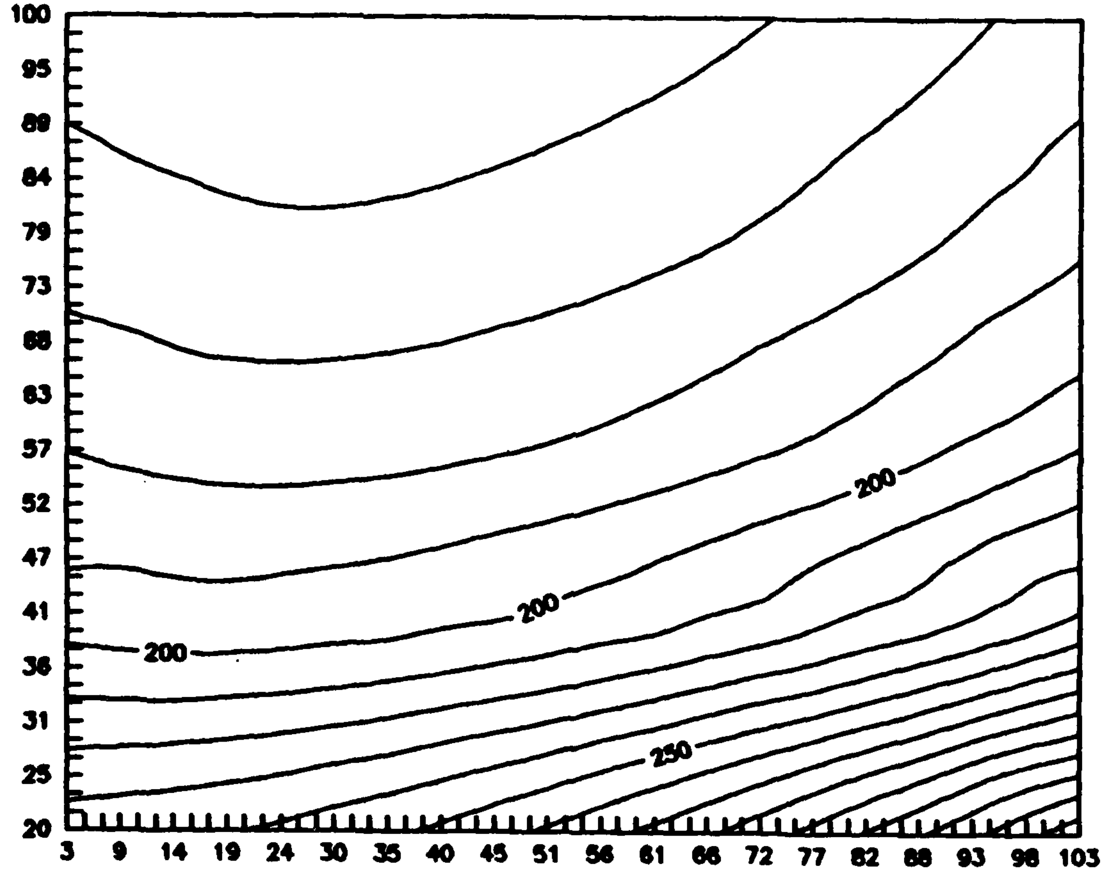

Fig E.44: Periapsis Contour (= 0 deg)

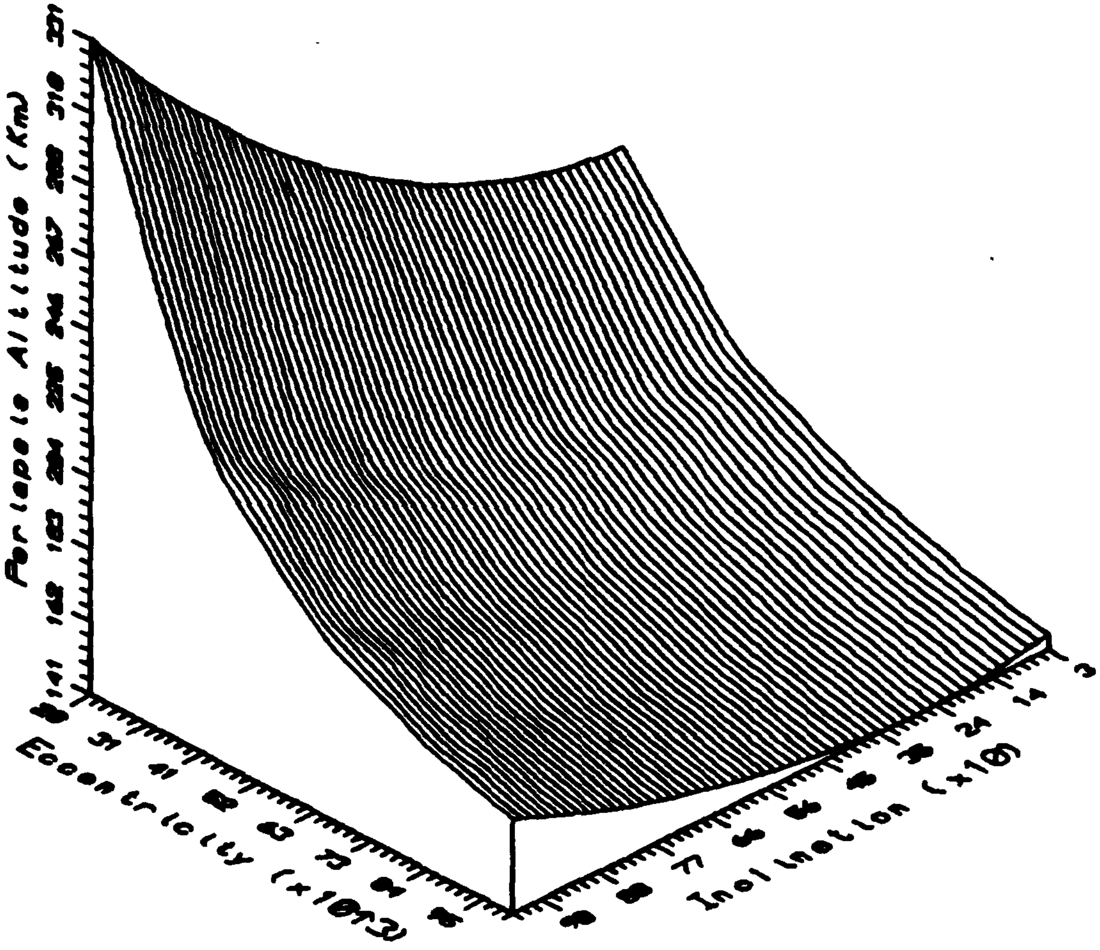

Fig E.45: Periapsis Altitude (= 15 deg)

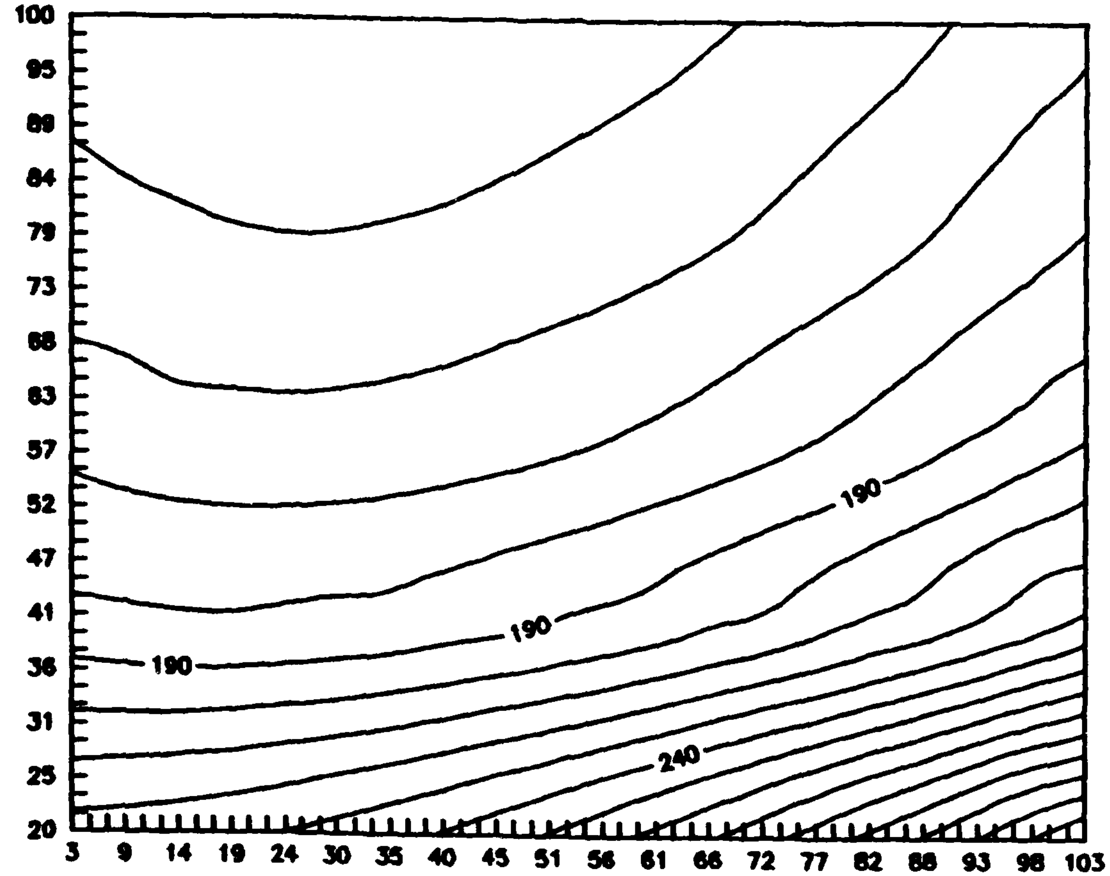

Fig E.46: Periapsis Contour (= 15 deg)

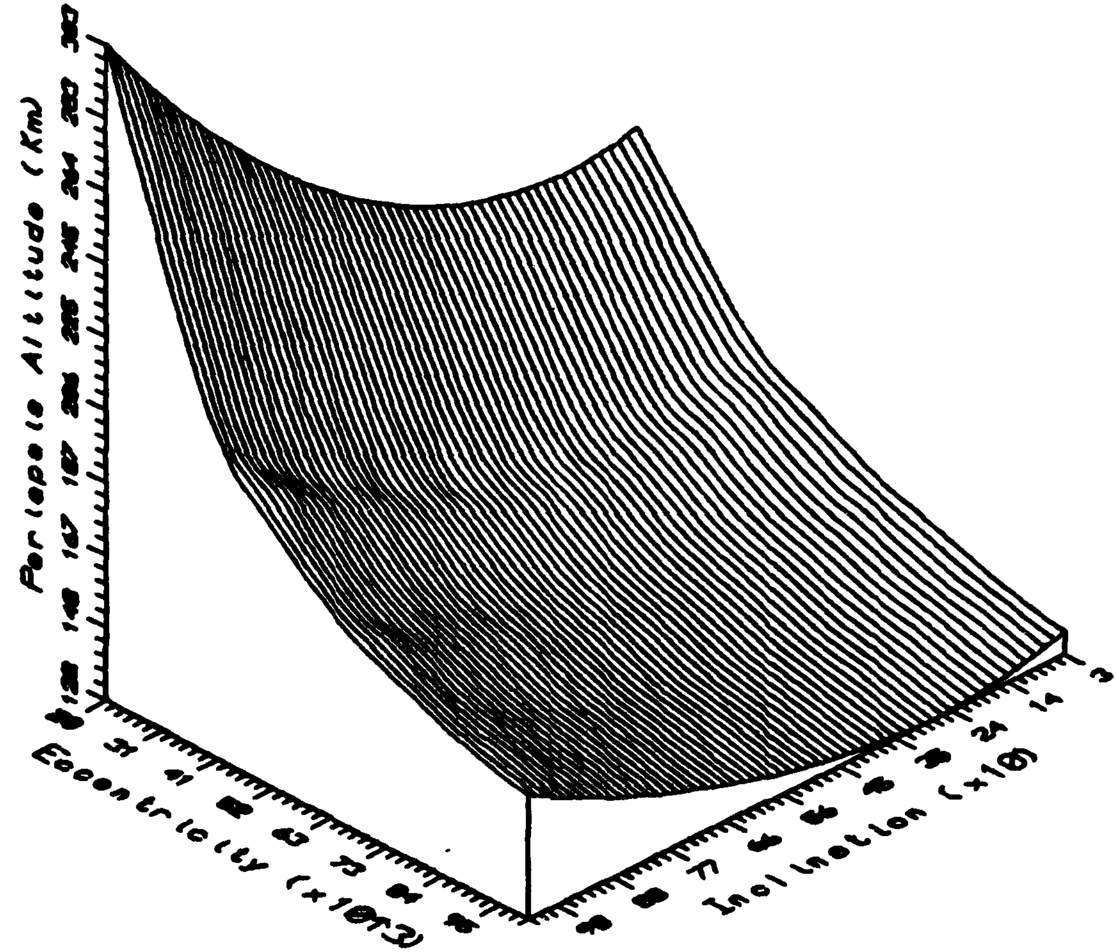

Fig E.47: Periapsis Altitude (= 60 deg)

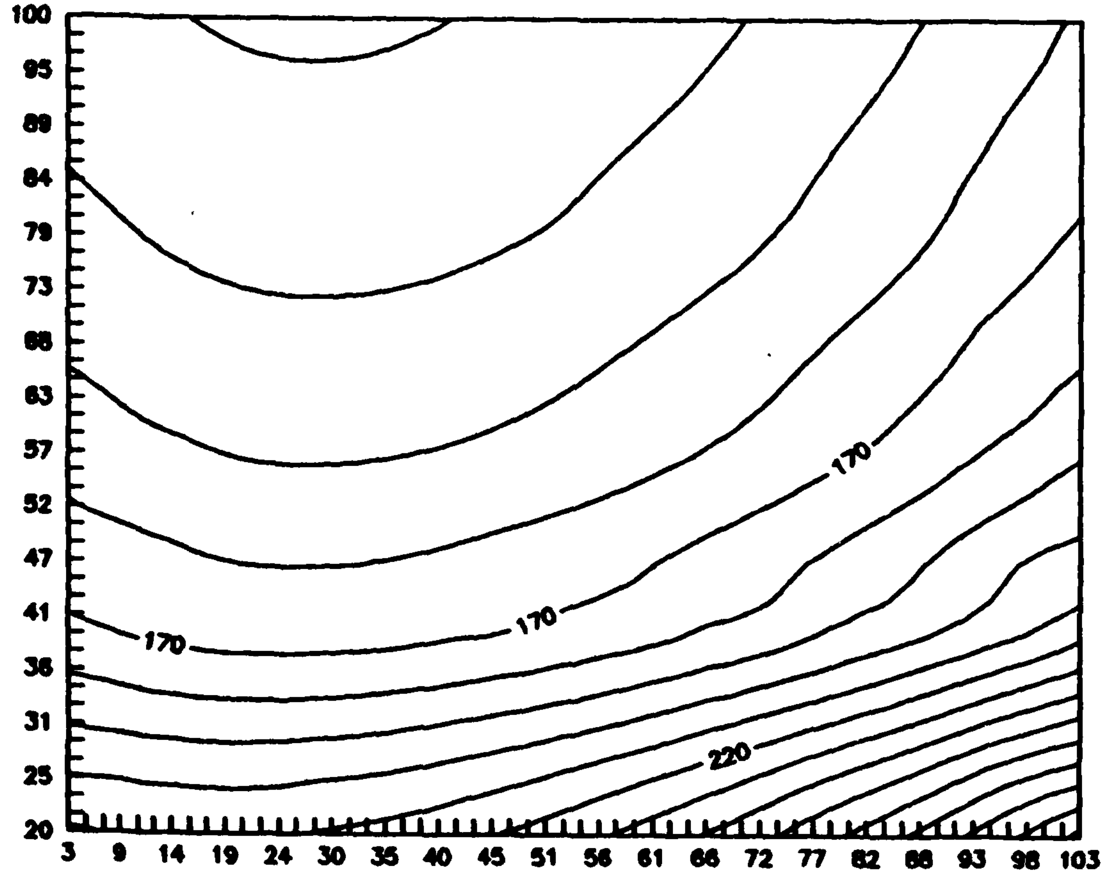

Fig E.48: Periapsis Contour (= 60 deg)

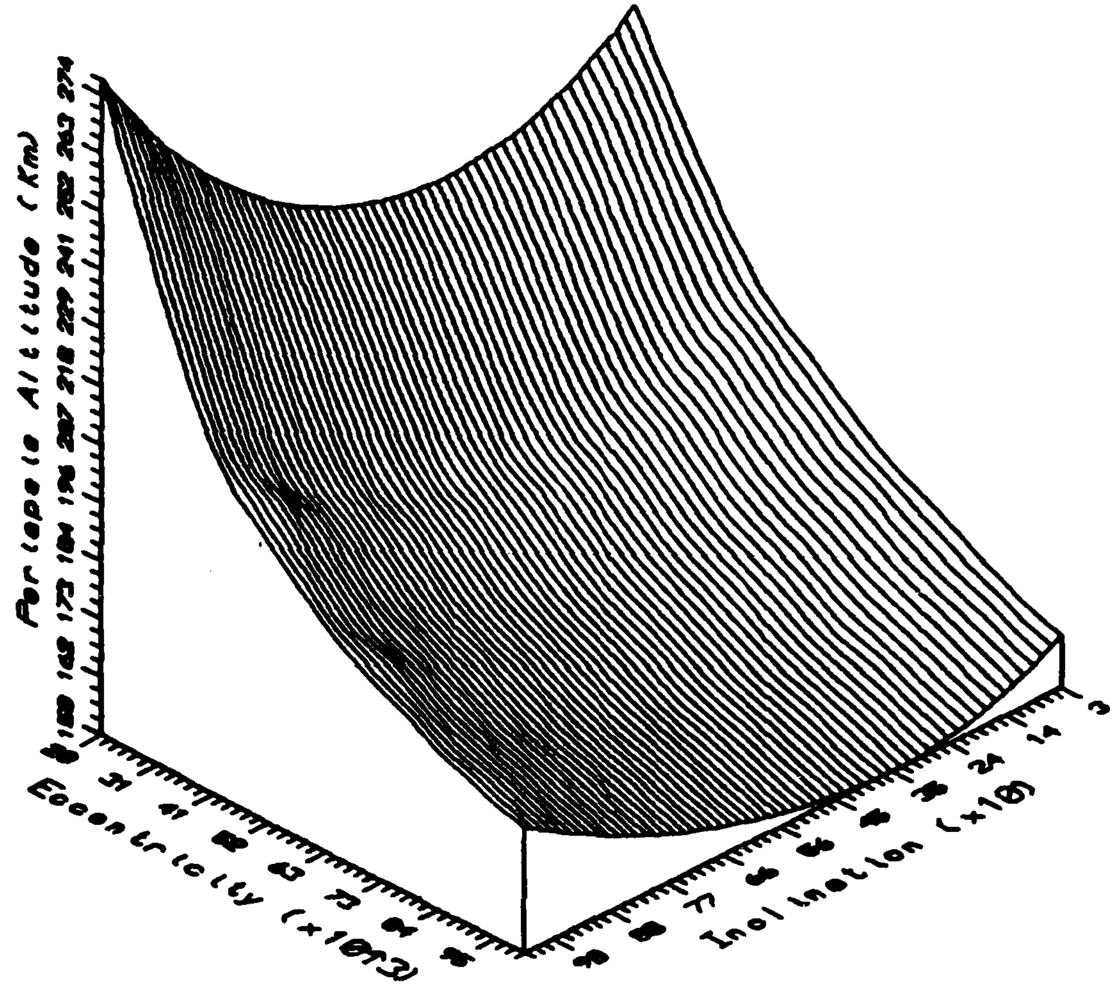

Fig E.49: Periapsis Altitude (= 105 deg)

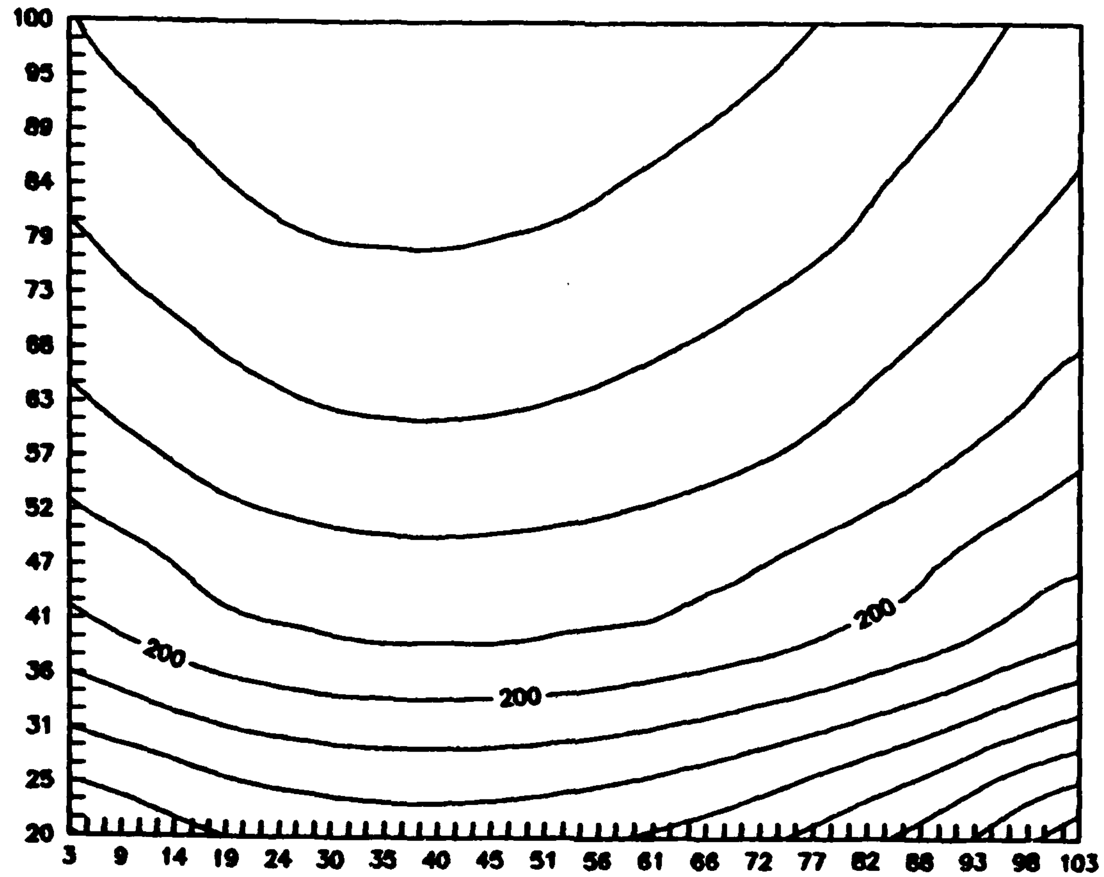

Fig E.50: Periapsis Contour (= 105 deg)

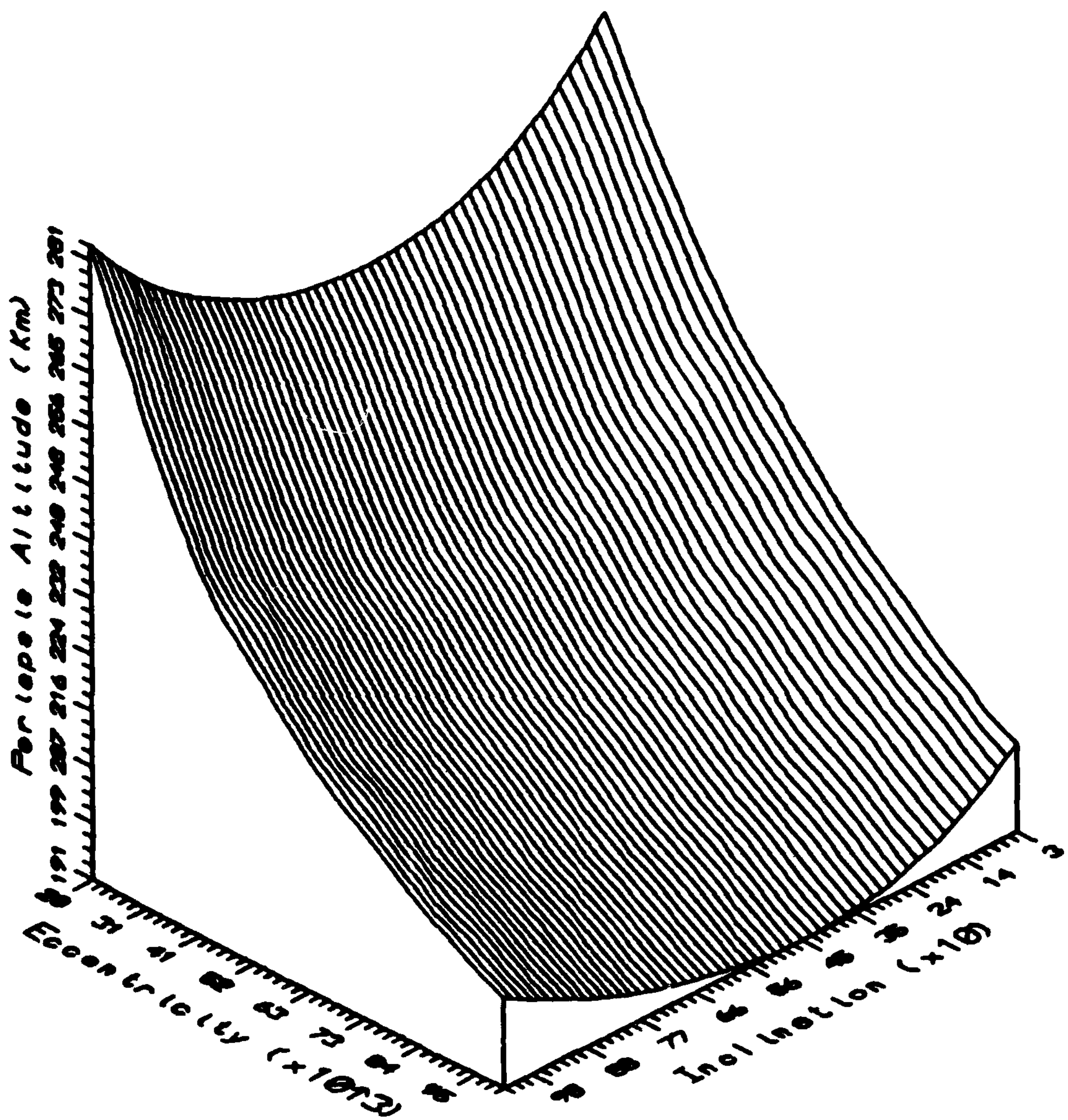

Fig E.51: Periapsis Altitude (= 150 deg)

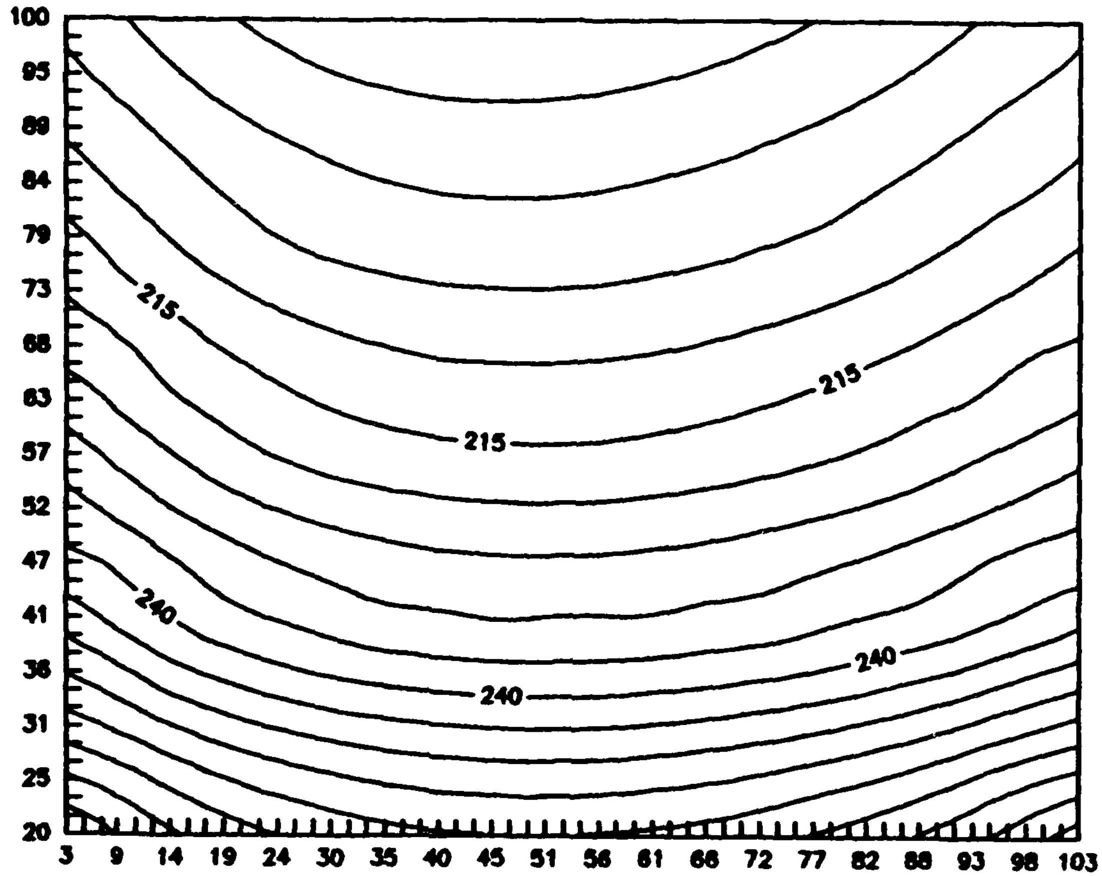

Fig E.52: Periapsis Contour (= 150 deg)

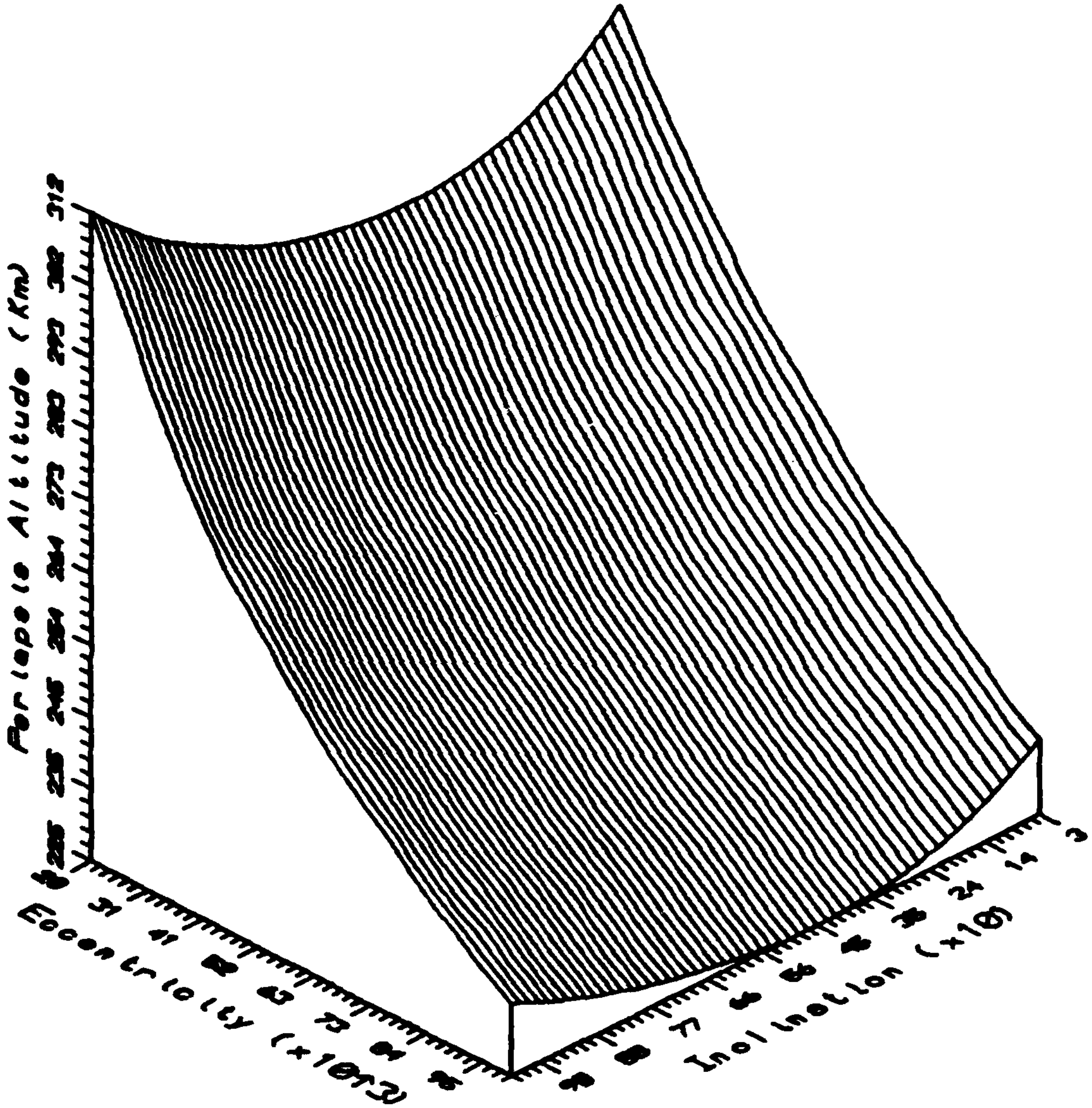

Fig E.53: Periapsis Altitude (= 195 deg)

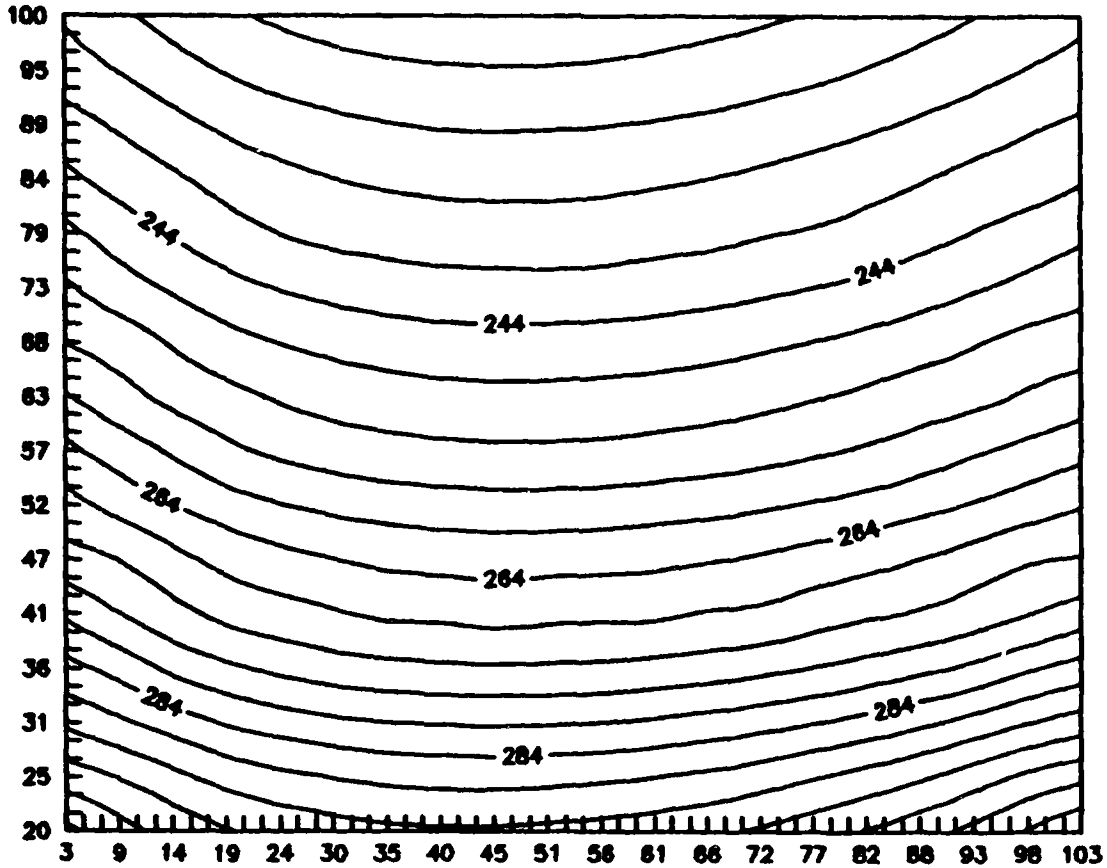

Fig E.54: Periapsis Contour (= 195 deg)

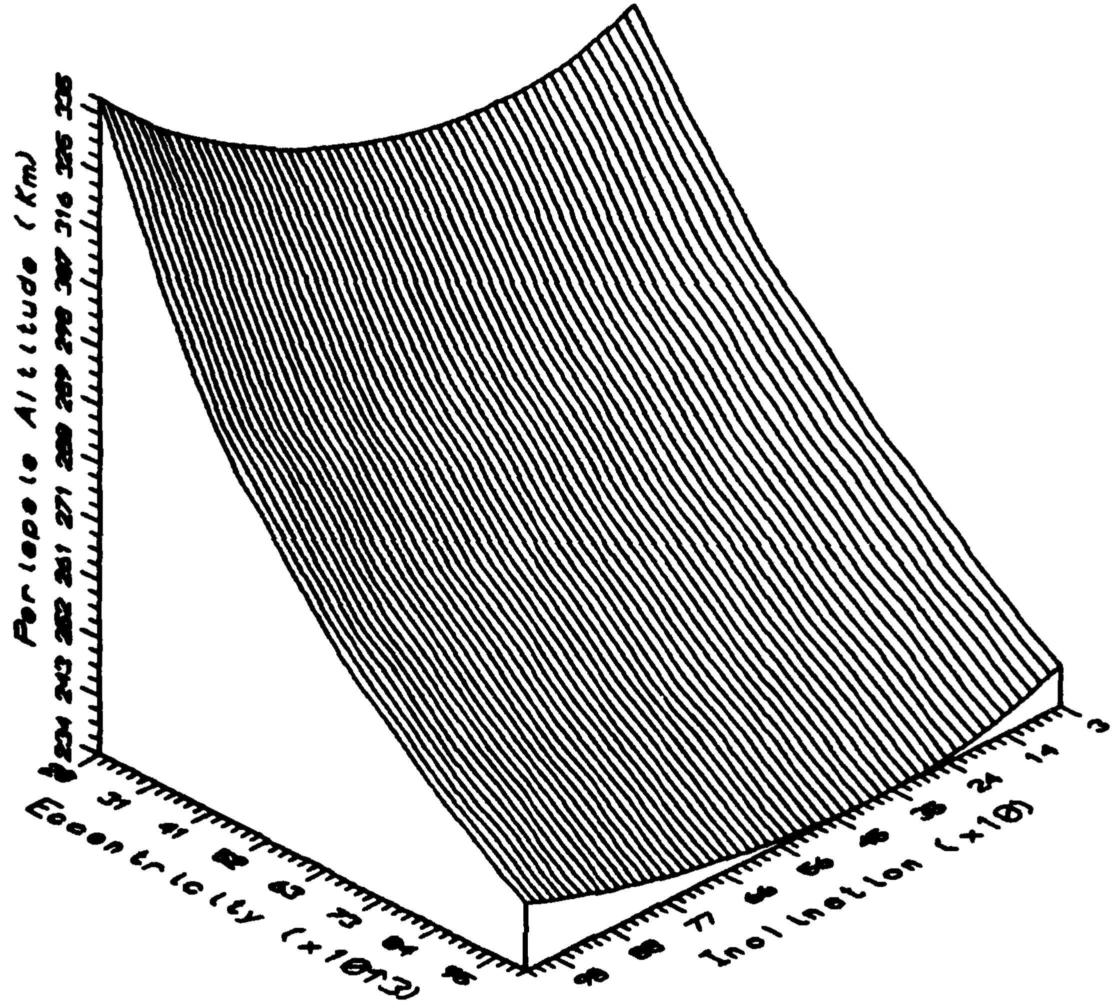

Fig E.55: Perlapsis Altitude (= 240 deg)

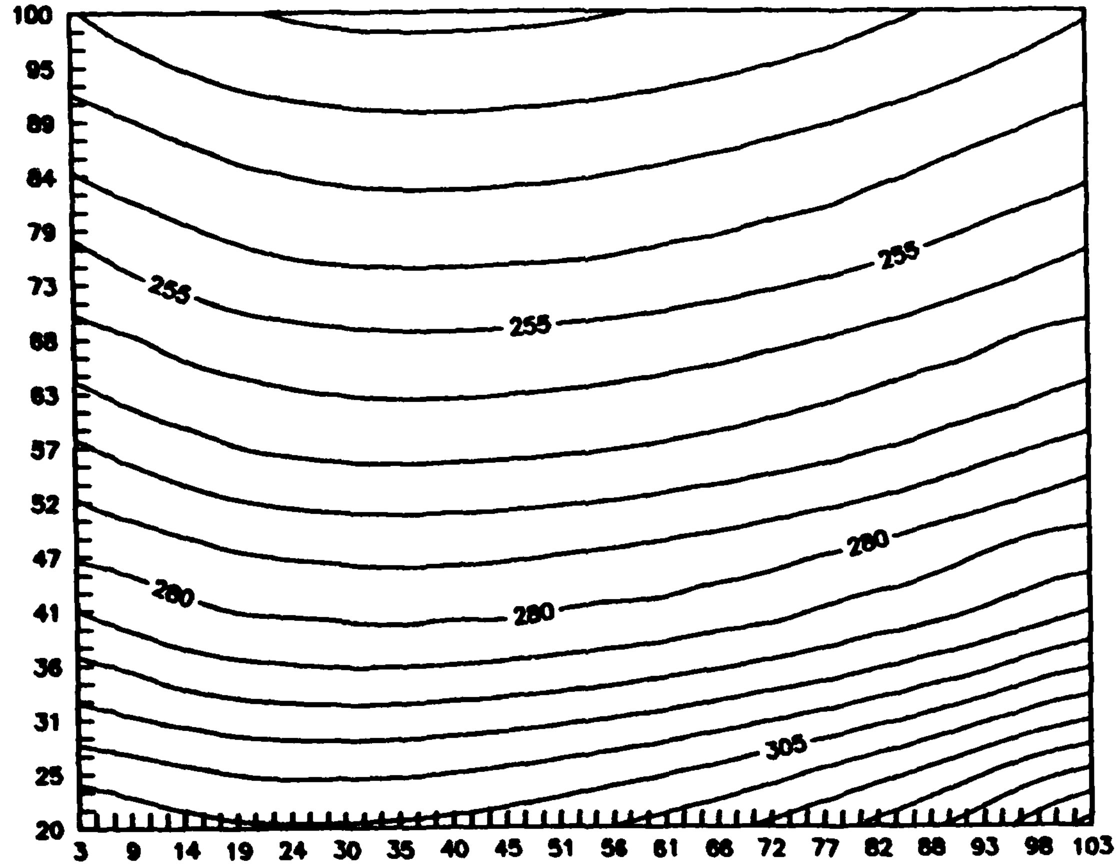

Fig E.56: Periapsis Contour (= 240 deg)

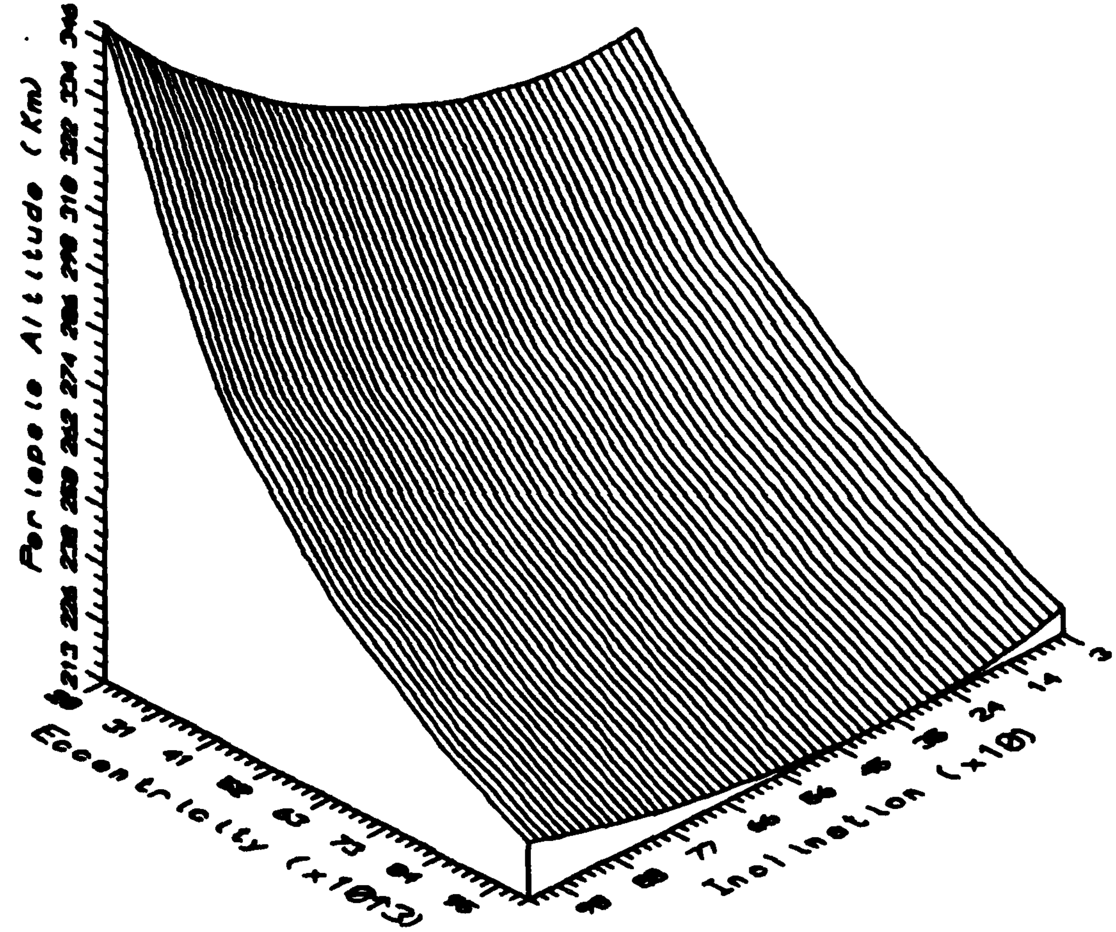

Fig E.57: Periapsis Altitude (= 285 deg)

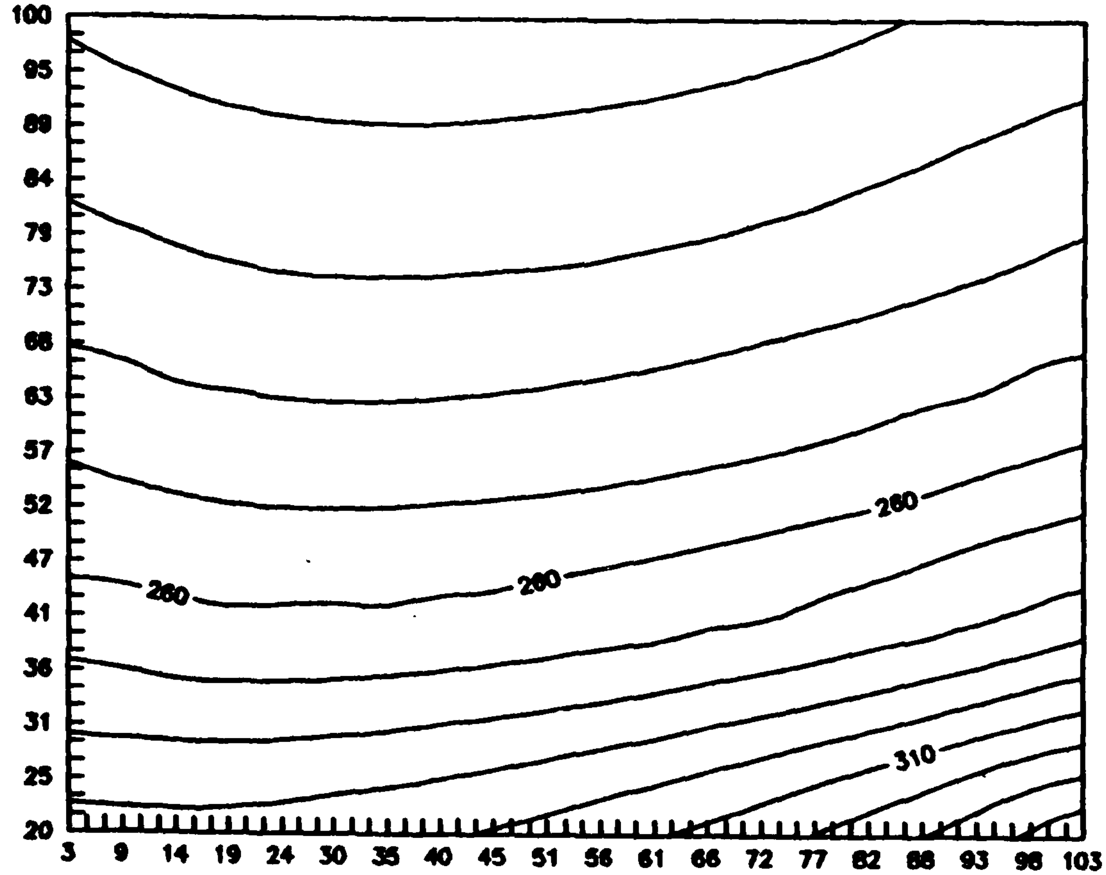

Fig E.58: Periapsis Contour (= 285 deg)

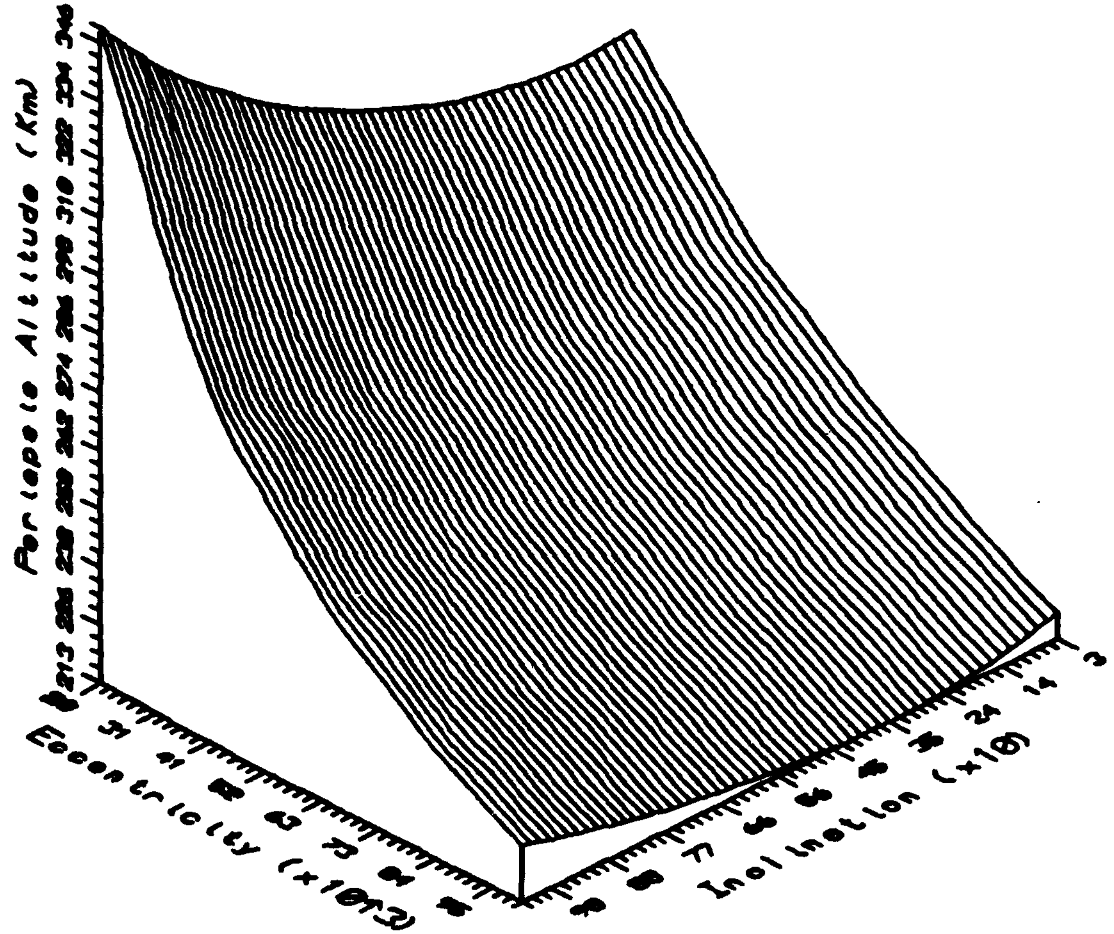

Fig E.59: Periapsis Altitude (= 330 deg)

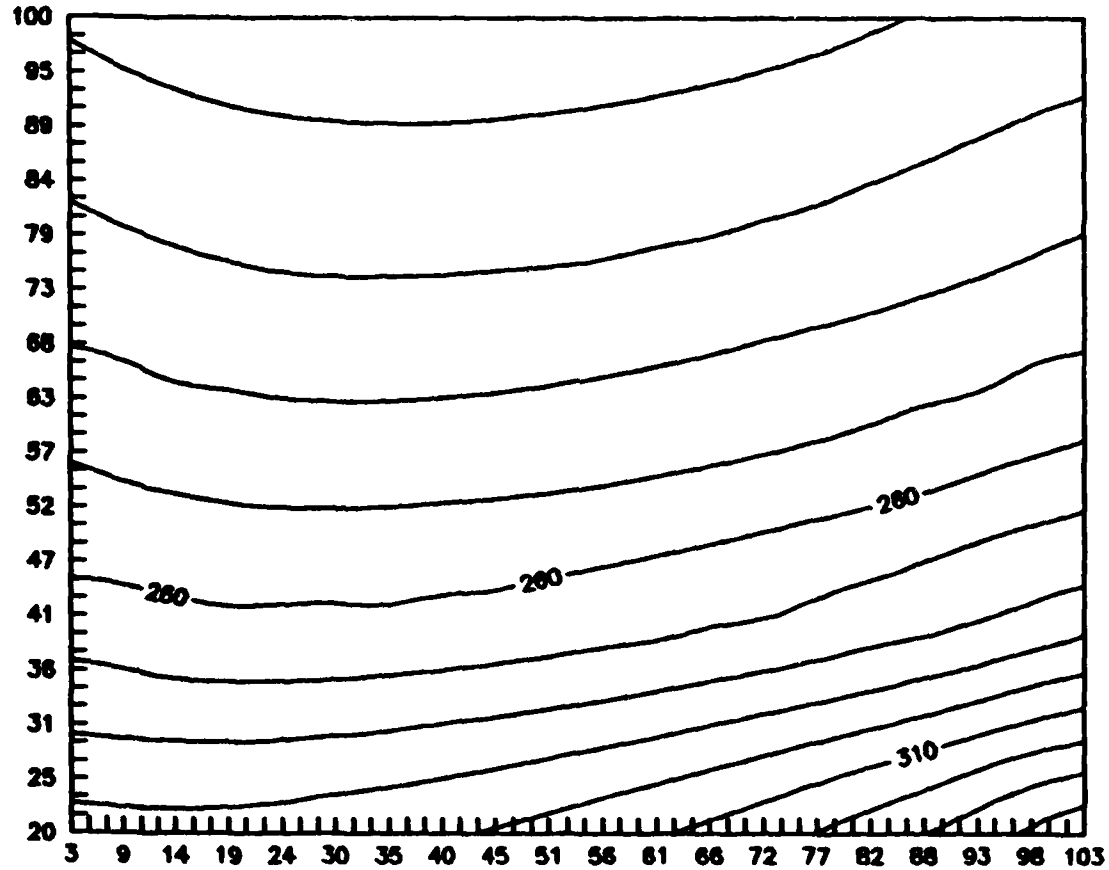

Fig E.60: Periapsis Contour (= 330 deg)

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Vita

Captain Robert L. Dudley, Jr. was born on [redacted] In 1975 he left high school and joined the army. During his first tour of duty in Korea, he finished his high school education and met his wife-to-be, [redacted] After recieving an honorable discharge from the army, he attended Oakland University where he recieved a B.S. in physics in December 1982. Upon graduation from Oakland University, he attended Officer Training School and accepted a commission in the United States Air Force. His first assignment was to attend Parks College of Saint Loius University in persuit of a B.S. in Aeronautical Engineering. He graduated Magna Cum Laude at the head of his class in December 1985. His next assignment took him to the Ballistic Missile Office, Norton AFB, California where he worked on the qualification of the Mk21 Reentry Vehicle for use on the Peacekeeper Missile. He stayed at the Ballistic Missile Office for two years whereupon he was selected for attendence in the School of Engineering, Air Force Institute of Technology, in April 1985.

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Thesis Chairman: Rodney D. Bain, Captain, USAF
Instructor of Astronautical Engineering

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Block 19:

This study develops an analytic function for the six dimensional surface (or hypersurface) above the planet Venus for a five earth year survivability for an artificial satellite.

Current US policy concerning the exploration of other planets, via artificial satellites, requires the satellites be sterilized (5:61). This is a very time intensive and costly practice. Developing the ability to estimate the life time of an artificial satellite that can no longer perform its station keeping duties may allow the sterilization procedures before launch to be waived. The objective is to develop a five year survivability function ~~(denoted by h_p)~~ in the orbital parameter space above which a satellite has at least five years to survive before it impacts the planet's surface.

Perturbations effects which would cause the satellite's orbit to deteriorate are modeled and include: a) the geopotential of the planet, b) the effects of solar wind, and c) the drag on the satellite due to the atmosphere of the planet.

The model was then interpolated to provide an analytic function for five year survivability. Keywords: [illegible] field 19